

A preliminary univalent formalization of the p -adic numbers

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Abstract

The goal of this paper is to give a preliminary formalization of the p -adic numbers, in the context of the Univalent Foundations. We also provide the corresponding code verifying the construction in the proof assistant Coq. Because work in the univalent setting is ongoing, the structure and organization of the construction of the p -adic numbers we give in this paper is expected to change as Coq libraries are more suitably rearranged, and optimized, by the authors and other researchers in the future. So our construction here should be deemed as a first approximation which is subject to improvements.

1 Introduction

In this paper we present a preliminary formalization of the construction of the p -adic numbers in the Coq proof assistant. The formalization is carried out in the *univalent setting* introduced by the second author [19]. This setting, which is based on insights from homotopy theory and higher-dimensional category theory, serves as an overall organizational and methodological framework which informs our construction. At the same time, our construction has several ingredients which are familiar in constructive mathematics. Because work on formalization in this direction is ongoing, the Coq code associated with this paper may be updated accordingly in the future by the authors and others. As such, the structure and content of the Coq code described here may not match exactly the code which is ultimately included in the Univalent Foundations libraries. Readers interested in making use of the code should accordingly consult the latest version available.

We chose to formalize the p -adic numbers as a first step in the development and formalization of the p -adic theory of integrable systems. We hope that this will prove to be a promising approach to this theory which should facilitate progress in the field in the future, in particular with regard to the construction of algorithms and their numerical analysis. Ultimately, we hope that insights from this project could be useful in the setting of real integrable systems.

The idea of the univalent perspective is, roughly, to develop mathematics within the world of homotopy types. By virtue of taking this approach we are able to make use of type theory as a calculus for formal reasoning about homotopy types. We hope that in the

future, because this development of mathematics can be carried out in a proof assistant such as Coq so that the proofs carry some algorithmic content, it will be possible to extract good algorithms from the proofs. One of our motivations is that the construction of such algorithms would in turn help with some problems concerning integrable systems which are of particular interest in applications. For instance, one outstanding problem is: given numerical spectral data about a quantum system (coming from an experiment), extract an algorithm to reconstruct the classical integrable system, see Section 7.

We will only briefly touch upon the technical details of homotopy type theory and the univalence axiom, and we refer the reader to [2] for a basic introduction to homotopy type theory. For univalent foundations and the second author’s Coq library [18] we refer readers to [15], where a description of the research program, its motivations, and its implementation in Coq, are given. Because it is assumed that the reader is already familiar with Coq and with the second author’s program, this paper has been written in a style which we foresee future papers in formalization taking: it is a summary of the Coq code written in ordinary mathematical English. The details are of course in the Coq code, but the overall structure of the formalization (as well as the key steps of the proofs) should be apparent from the sketch given here. The actual Coq code associated to this paper can be found on the websites of the authors, as supplementary files to the arXiv posting of this paper, and as an appendix to the present paper.

Structure of paper

Hensel [10] invented the p -adic numbers \mathbb{Q}_p about one hundred years ago. The p -adic numbers and the reals are the canonical metric completions of the rationals. Classically, there are a number of ways to construct the p -adic numbers, and we refer the reader to [8, 11, 16] for further details regarding the classical theory. The construction of the p -adic numbers given in this paper is constructive and uses algebraic, rather than analytic, techniques. Namely, we first construct the integral domain of p -adic integers \mathbb{Z}_p as a quotient of the ring $\mathbb{Z}[[X]]$ of formal power series over \mathbb{Z} . We were unable to find the specific construction of \mathbb{Z}_p we employ in the literature, but we believe that it is known. We then take the p -adic numbers \mathbb{Q}_p to be the field of fractions of \mathbb{Z}_p . Because we are working constructively, and because $\mathbb{Z}[[X]]$ does not have decidable equality, it is necessary to work with an apartness relation and with the corresponding notions of integral domains and fields. We will refer to the apartness versions of fields as *Heyting fields* following the standard usage in constructive mathematics.

In detail, this paper is organized as follows. In Section 2, we give a brief overview of the univalent setting. In Section 3 we review some basic constructive algebra. Section 4 contains our construction of formal power series and the proofs of several results on formal power series. The proof that it is possible to form the Heyting field of fractions for an integral domain is given in Section 5. The construction of the p -adic numbers appears in Section 6. Section 7 is a brief epilogue containing a sketch of some future plans concerning p -adic integrable systems. Finally, the Coq code can be found in the Appendix 7.2. Although this appendix is quite long, it is the most important part of the paper and so we feel that it is justified to include it here.

We should note that the p -adic numbers are also relevant in the physics literature, see [5] and the references therein. In fact, one of our main motivations in wanting to develop a p -adic theory of integrable systems is to study inverse spectral problems concerning p -adic analogues of real quantum integrable systems. We refer to Section 7.2 for a list of short term plans concerning the p -adic numbers.

2 Univalent basics

The second author's Coq library span a large portion of mathematics and we make free use of this library. However, for the sake of clarity we will here mention those specific parts of the library which we use in the construction of the p -adic numbers. A survey of the development of univalent mathematics in Coq can be found in [15].

Notation and conventions

In this paper, and in the Coq files, all rings are assumed to be commutative and with 1.

\mathbb{N} denotes the type of natural numbers which is defined as an inductive type in the standard way. In the Coq code \mathbb{N} is denoted by `nat`. Similarly, \mathbb{Z} denotes the type of integers which is constructed as the group completion of the abelian monoid of natural numbers. In the Coq code \mathbb{Z} is denoted by `hz`.

\mathcal{U} denotes a fixed universe of types. In the Coq code this is denoted by `UU`. The identity type $\text{Id}_A(a, b)$ is denoted by $a \rightsquigarrow b$. In the Coq files this is denoted by either `paths a b` or by $a \rightsquigarrow b$.

We write $\prod_{x:A} .B(x)$ for dependent products and $\sum_{x:A} .B(x)$ for dependent sums (defined here as the record type `total2`).

We will generally use the same naming conventions as used in the Coq files, but in some cases we will introduce abbreviations, such as $\sum_{i=0}^n f(i)$ for summation, when it will improve the readability.

Because the current implementation of the underlying type system of Coq does not handle universes (and several related matters) in a way which is completely suited for the univalent development of mathematics, it is necessary to apply several patches to the Coq system in order to compile the second author's Coq library as well as the files described in this paper. Instructions on how to compile a patched version of Coq can be found in the second author's library.

2.1 Basic homotopy theoretic notions in Coq

We think of \mathcal{U} as the universe of small homotopy types (or fibrant and cofibrant spaces). For $B : \mathcal{U}$, we represent a dependent type over B as a term $E : B \rightarrow \mathcal{U}$. From the perspective of homotopy theory this corresponds to a fibration over B and, for $b : B$, $E(b)$ corresponds to the fiber over b . The dependent product $\prod_{x:B} E(x)$ is regarded as the space of sections of the fibration represented by E . Similarly, the dependent sum, $\sum_{x:B} E(x)$ corresponds to

the total space of the fibration. We think of the identity type $a \rightsquigarrow b$ as denoting the fiber of the path space over (a, b) . We will use the phrases "path space" and "type of paths" interchangeably for this type. I.e., a term $f : a \rightsquigarrow b$ corresponds to a path from a to b .

Given a path $f : b \rightsquigarrow b'$ in B and a point $e : E(b)$ in the fiber over b we obtain a corresponding point $f_!(e) : E(b')$ in the fiber over b' . In the Coq code $f_!$ is denoted by `transportf E f e`. In order to construct a path $x \rightsquigarrow y$ in the total space $\sum_{x:B} E(x)$ it suffices to construct a path $f : \pi_1(x) \rightsquigarrow \pi_1(y)$ and a path $g : f_!(\pi_2(x)) \rightsquigarrow \pi_2(y)$.

Given a term $g : B \rightarrow A$ and a path $f : b \rightsquigarrow b'$ in B , we obtain a path $g(f) : g(b) \rightsquigarrow g(b')$. In the Coq code $g(f)$ is denoted by `maponpaths g f`. This corresponds, regarding a homotopy type as an ∞ -groupoid, the weakly functorial action of g on the path f .

Definition 2.1 (`hfiber`). Given types A and B , $g : B \rightarrow A$ and $a : A$, the **homotopy fiber of g over a** is the type

$$\mathbf{hfiber} \ g \ a := \sum_{x:B} (g(x) \rightsquigarrow a).$$

Definition 2.2 (`iscontr`). We define the type **`iscontr`**(A) of proofs that A is contractible as

$$\mathbf{iscontr}(A) := \sum_{c:A} \prod_{x:A} (x \rightsquigarrow c).$$

We say that A is **contractible** if **`iscontr`**(A) is inhabited.

We will see below that contractibility in this setting plays the same role as canonical existence in the classical development of mathematics.

Definition 2.3 (`isweq` and `weq`). Given $g : B \rightarrow A$ we define the type **`isweq`**(g) of proofs that g is a weak equivalence as

$$\mathbf{isweq}(g) := \prod_{x:A} \mathbf{iscontr}(\mathbf{hfiber} \ g \ x).$$

If **`isweq`**(g) is inhabited, then we say that g is a **weak equivalence**.

There is a filtration of types into different "h-levels". Homotopy theoretically this is a slight extension of the usual filtration by homotopy n -types. We will only require the first few h-levels in this paper.

Definition 2.4 (`isofhlevel`, `isaprop`, `hprop`, `isaset` and `hset`). A type A is of **h-level**:¹

- 0 if A is contractible;
- $(n + 1)$ if, for all $a, b : A$, the type $(a \rightsquigarrow b)$ is of h-level n .

¹Note that in order to define `isofhlevel` as a type which has values in \mathcal{U} , as is done in the file `uu0.v` from the second author's Coq library, it is necessary to compile Coq with a patch.

We denote by $\iota_n(A)$ the type of proofs that A is of h-level n . We abbreviate $\iota_1(A)$ by **isaProp**(A) and $\iota_2(A)$ by **isaSet**(A). We write **hProp** for the type of (small) types of h-level 1 and **hSet** for the type of (small) types of h-level 2.

Intuitively, **hProp** consists of those spaces which are homotopy equivalent to either the empty space 0 or to the one element space 1. Accordingly, **hProp** plays the role played by the Booleans in classical logic or by the subobject classifier in topos logic. Types in **hProp** satisfy proof-irrelevance (**proofirrelevance**) and, indeed (**invproofirrelevance**), being an h-prop is equivalent to being proof-irrelevant.

Intuitively, **hSet** consists of those spaces which are homotopy equivalent to discrete spaces. I.e., these are the sets. Most of the types which we will be dealing with are either h-props or h-sets. We will sometimes refer to h-sets simply as "sets" when no confusion will result.

We make use of a number of basic properties of h-levels. E.g.,

1. **impred**: for $n : \mathbb{N}$, $B : \mathcal{U}$ and $E : B \rightarrow \mathcal{U}$, the type

$$\prod_{x:B} \text{isofhlevel}_n(E_x) \rightarrow \text{isofhlevel}_n(\prod_{x:B} E_x)$$

is inhabited.

2. **impredfun**: for $n : \mathbb{N}$, $A, B : \mathcal{U}$, if A is of h-level n , then so is $(B \rightarrow A)$.
3. **isofhleveldirprod**: If A is of h-level n and B is of h-level n , then so is $A \times B$.

2.2 Function extensionality

We make extensive use of the principle of function extensionality (**funextfun**), which follows from the second author's *Univalence Axiom*.

Definition 2.5 (**funextfun**). The principle of **function extensionality** states that, for any two functions $f, g : A \rightarrow B$, the type

$$\left(\prod_{x:A} f(x) \rightsquigarrow g(x) \right) \rightarrow (f \rightsquigarrow g)$$

is inhabited.

2.3 Properties of hProp

Given a type $A : \mathcal{U}$, there is a universal way to turn A into a h-prop. This is the "inhabited" construction:

Definition 2.6 (`ishinh_ℳ`). We say that $A : \mathcal{U}$ is **h-inhabited** if the type

$$\hat{A} := \prod_{P : \mathbf{hProp}} ((A \rightarrow P) \rightarrow P)$$

is inhabited.

It is immediate, using the facts about h-levels sketched above to see that \hat{A} is an h-prop. Moreover, there is a projection $\pi_A : A \rightarrow \hat{A}$ given by

$$\pi_A := \lambda_{x:A} \cdot \lambda_{P:\mathbf{hProp}} \cdot \lambda_{f:A \rightarrow P} \cdot f(x).$$

The map π_A is the universal map from A into a h-prop. To see this, observe that if Q is any h-prop and $f : A \rightarrow Q$, then we have a commutative (up to definitional equality) diagram

$$\begin{array}{ccc} \hat{A} & \xrightarrow{\bar{f}} & Q \\ \pi_A \swarrow & & \searrow f \\ & A & \end{array}$$

where

$$\bar{f} := \lambda_{t:\hat{A}} \cdot t(Q)(f).$$

Moreover, since Q is a h-prop it follows (using function extensionality) that the space of such extensions \bar{f} is contractible.

Using the h-inhabited construction it is possible to endow **hProp** with the structure of a Heyting algebra. This structure is summarized below:

Definition 2.7 (`htrue`, `hfalse`, `hconj`, `hdisj`, `hneg`, `himpl`). For $P, Q : \mathbf{hProp}$ and $X, Y : \mathcal{U}$ we define logical operations on **hProp** as follows:

- 1 and 0 are h-props.
- $P \wedge Q := P \times Q$.
- $X \vee Y := \widehat{X + Y}$.
- $\neg X := X \rightarrow 0$.
- $X \implies P := X \rightarrow P$.

In addition to the Heyting algebra operations, there is an existential quantifier (`hexists`) which is defined by

$$\exists_{x:X} P(x) := \widehat{\sum_{x:X} P(x)}$$

for any $P : X \rightarrow \mathcal{U}$ and $X : \mathcal{U}$. This quantifier satisfies the usual properties of the existential quantifier in intuitionistic logic. Note that our \exists does *not* correspond to the built-in existential quantifier ”**exists**” in Coq.

The proof that, with the operations above, **hProp** is a Heyting algebra makes use of the *Propositional Univalence Axiom* (**uahp**) which says that every logical equivalence between h-props induces a path between them. I.e., it says that the type

$$\prod_{P, Q : \mathbf{hProp}} (P \rightarrow Q) \rightarrow ((Q \rightarrow P) \rightarrow (P \rightsquigarrow Q)).$$

is inhabited.

2.4 Set quotients of types

The second author has given several constructions of quotients of types. A **hsubtype** of a type A is given by a map $S : A \rightarrow \mathbf{hProp}$. Denote by $\mathcal{P}(A)$ the type of hsubtypes of A . Given a relation R on A (that is, $R : A \rightarrow A \rightarrow \mathbf{hProp}$), an **equivalence class** consists of a subtype S of A together with the following data:

1. a term of type $\widehat{\sum_{x:A} S(x)}$.
2. a term of type $\prod_{x,y:A} (xRy \rightarrow S(x) \rightarrow S(y))$.
3. a term of type $\prod_{x,y:A} (S(x) \rightarrow S(y) \rightarrow xRy)$.

Given a subtype S , we denote by $\mathbf{iseqclass}_R(S)$ the type consisting of such data. The **set quotient** A/R (**setquot**) of a type A by a relation R is then defined by

$$A/R := \sum_{S : \mathcal{P}(A)} \mathbf{iseqclass}_R(S).$$

It is shown (**isasetsetquot**) in the second author’s library that A/R is a set and that, when R is an equivalence relation, this set has the usual universal property. In particular, there is a function $\pi : A \rightarrow A/R$ (**setquotpr**) which is compatible with the equivalence relation and, for any set B and function $f : A \rightarrow B$ which is compatible with R , there exists an extension \bar{f} making the diagram

$$\begin{array}{ccc} A/R & \xrightarrow{\quad \bar{f} \quad} & B \\ & \swarrow \pi \quad \searrow f & \\ & A & \end{array}$$

commute. We will make free use throughout of the results on set quotients from the second author’s library.

3 Basics on constructive algebra

We will here briefly recall some basics of constructive algebra. For a more detailed treatment we refer to [6] and [12].

The usual definitions of fields and integral domains are not entirely satisfactory from the perspective of constructive algebra since they deal with negative properties (the property of being a non-zero element of the field). From the constructive perspective, it is more appropriate to replace the notion of an element x being non-zero ($x \neq 0$) with x being **apart from zero**, written $x \# 0$.

We will now recall the basics regarding apartness relations.

Definition 3.1. (*isapart*) A relation $R : \mathbf{hRel}(X)$ is an **apartness relation** provided that it satisfies the following conditions:

Irreflexive for all $x : X$, $\neg(xRx)$.

Symmetric for all $x, y : X$, xRy implies yRx .

Cotransitive for all $x, y : X$, if xRy , then either xRz or zRy , for any $z : X$.

Classically, the negation of equality $x \neq y$ relation is an apartness relation. However, negation of equality is not the only classical apartness relation. For example, if X is a topological space, then the relation R given by xRy if and only if x and y are in different connected components is an apartness relation. (This example can be generalized to give a limitless number of classical examples of apartness relations.)

For $X : \mathbf{hSet}$, we denote by **Apart**(X) the type of apartness relations on X . We generally denote apartness relations by $x \# y$. When a type has decidable equality the negation of equality is an apartness relation:

Lemma 3.2 (*deceqtoneqapart*). *If $X : \mathbf{hSet}$ has decidable equality, then negation of equality*

$$\neg(x \approx y)$$

is an apartness relation on X .

Definition 3.3 (*isapartdec*). Given $X : \mathbf{hSet}$ and $R : \mathbf{Apart}(X)$, we say that R is a **decidable apartness relation on X** if the type

$$(aRb) + (a \approx b)$$

is inhabited.

It is immediate (*isapartdectodeceq*) that if R is a decidable apartness relation on X , then X has decidable equality.

When we are considering algebraic structures equipped with apartness relations we will require that the relation is compatible with the operations under consideration. In particular, for rings we have the following.

Definition 3.4 (`acommrng`). The type **aCRng** consists of commutative rings A together with an apartness relation $x \# y$ on A which is compatible with the ring structure of A in the sense that²

- (i) For all $a, b, c : A$, if $(c + a) \# (c + b)$, then $a \# b$.
- (ii) For all $a, b, c : A$, if $(c \cdot a) \# (c \cdot b)$, then $a \# b$.

When a commutative ring A has decidable equality it is straightforward to verify that negation of equality is compatible with the ring operations in the sense of Definition 3.4.

Definition 3.5 (`aintdom`). The type **aDom** consists of $A : \mathbf{aCRng}$ such that

- $1 \# 0$.
- For all $a, b : A$, if $a \# 0$ and $b \# 0$, then $(a \cdot b) \# 0$.

We refer to the terms of type **aDom** as **apartness domains**.

Heyting fields are the appropriate generalization of fields to the constructive setting when one considers algebraic structures with apartness relations:

Definition 3.6 (`afld`). The type **aFld** of **Heyting fields** consists of $A : \mathbf{aCRng}$ such that

- $1 \# 0$.
- For all $a : A$, if $a \# 0$, then a has a multiplicative inverse (the type of multiplicative inverses of a is inhabited).

We have the following immediate observation:

Lemma 3.7 (`afldtoaintdom`). *If A is a Heyting field, then A is an apartness domain.*

Proof. It is immediate to prove that, in a Heyting field, if a has a multiplicative inverse, then it is apart from 0 (`afldinvertibletoazero`). It follows that $1 \# 0$. One can show that if a and b both possess multiplicative inverses, then so does their product $a \cdot b$ (`multinvmultstable`). It is then immediate that $(a \cdot b) \# 0$ when $a \# 0$ and $b \# 0$. \square

4 Formal power series

Our treatment of formal power series makes use of function extensionality, since formal power series over a commutative ring R are here defined as terms of type $\mathbb{N} \rightarrow R$ with the operations of addition and multiplication given in the usual way. The main result of this section is that, with these operations, formal power series is a commutative ring. Moreover, there is a natural apartness relation on formal power series and, furthermore, when the ring R has decidable equality the ring of formal power series over R forms an apartness domain. We will now fill in the details of this sketch.

²Note that in the Coq files we actually require the corresponding cancellation properties also on the right. This is redundant for commutative rings, but for general rings one requires also these further properties.

4.1 Summation in a ring

We define both a restrictive summation operation (`natsummation0`), which allows us to form the sum $\sum_{i=0}^n a_i$ of a sequence $a : \mathbb{N} \rightarrow R$, and a more general operation (`summation`), which allows us to form the sum $\sum_{i=m}^n a_i$ of a sequence $a : \mathbb{Z} \rightarrow R$. However, we will only really require the former of these two constructions and so we will omit details related to the more general summation. In order to avoid confusion with our notation for dependent sums, we write $\bigoplus_{i=0}^n a_i$ for the sum $\sum_{i=0}^n a_i$. Summation is, of course, defined inductively by setting

$$\bigoplus_{i=0}^0 a_i := a_0 \quad \text{and} \quad \bigoplus_{i=0}^{n+1} a_i := \left(\bigoplus_{i=0}^n a_i \right) + a_{n+1}.$$

Manipulation of sums

It is important to note that when we manipulate sums, to obtain new sums, *what is relevant is that there is a path between them, and not whether they are equal in the strict sense*. This is a crucial point which underlies in a fundamental way much of the univalent approach to mathematics. The following lemma includes several basic facts regarding the behavior of summation of which we will make frequent use:

Lemma 4.1. *Given a natural number n and sequences $a, b : \mathbb{N} \rightarrow R$, we have the following:*

1. (`natsummationpathsupperfixed`) *Given $p : \prod_{x:\mathbb{N}} (x \leq n) \rightarrow (a_x \rightsquigarrow b_x)$, the type*

$$\bigoplus_{i=0}^n a_i \rightsquigarrow \bigoplus_{i=0}^n b_i$$

is inhabited.

2. (`natsummationshift0`) *The type*

$$\bigoplus_{i=0}^{n+1} a_i \rightsquigarrow \left(\bigoplus_{i=0}^n a_{i+1} \right) + a_0$$

is inhabited.

In order to more easily handle reindexing of sums we introduce, for $f : \mathbb{N} \rightarrow \mathbb{N}$, the type **Aut** _{n} (f) (`isnattruncauto`) of proofs that f is an automorphism of the interval $[0, n]$ of natural numbers. Explicitly, **Aut** _{n} (f) is defined to be the following type:³

$$\left(\prod_{x \leq n} \sum_{y \leq n} ((f(y) \rightsquigarrow x) \times \prod_{z \leq n} (f(z) \rightsquigarrow x) \rightarrow (y \rightsquigarrow z)) \right) \times \left(\prod_{x \leq n} (f(x) \leq n) \right)$$

³Note that we could, alternatively, have used the type $(\prod_{x \leq n} \sum_{y \leq n} (f(y) \rightsquigarrow x)) \times (\prod_{x \leq n} (f(x) \leq n))$. However, the more verbose type we give here is convenient, for purposes of formalization, as it allows for more direct proofs of subsequent lemmas.

where we have abbreviated $\prod_{x:\mathbb{N}} (x \leq n) \rightarrow \cdots$ as $\prod_{x \leq n} \cdots$ and $\sum_{x:\mathbb{N}} (x \leq n) \times \cdots$ as $\sum_{x \leq n} \cdots$. It is possible to reindex sums along such automorphisms, as shown by the following lemma:

Lemma 4.2. (*natsummationreindexing*) *Given a natural number n and a map $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $\mathbf{Aut}_n(f)$ is inhabited, the type*

$$\bigoplus_{i=0}^n a_i \rightsquigarrow \bigoplus_{i=0}^n a_{f(i)}$$

for any sequence $a : \mathbb{N} \rightarrow R$, is inhabited.

The final fact regarding summation which we require is the following:

Lemma 4.3. (*natsummationswap*) *Given $f : \mathbb{N} \rightarrow \mathbb{N} \rightarrow R$ and a natural number n , the type*

$$\bigoplus_{k=0}^n \bigoplus_{l=0}^k f(l, k-l) \rightsquigarrow \bigoplus_{k=0}^n \bigoplus_{l=0}^{n-k} f(k, l)$$

is inhabited.

4.2 The ring of formal power series

We define, for a type A , the type of sequences of elements of A (**seqson**) as the function space $\mathbb{N} \rightarrow A$. When A is a set so is $\mathbb{N} \rightarrow A$ and for A a commutative ring we take $\mathbb{N} \rightarrow A$ as the underlying set (**fps**) of the ring of formal power series over A . If a is a sequence on A , then we write $a_n : A$ for the result of evaluating the sequence at the natural number n .

Ring operations on formal power series

For a given commutative ring R , addition and multiplication of formal power series are defined as usual by the formulae:

$$(a + b)_n := a_n + b_n$$

$$(a \cdot b)_n := \bigoplus_{k=0}^n a_k b_{n-k}.$$

The zero sequence 0 is given by $0_n := 0$ for all natural numbers n and the sequence 1 is given by $1_0 := 1$ and $1_{n+1} := 0$ for all natural numbers n .

Proposition 4.4 (**fpscommrng**). *Let $(R, +, \cdot)$ be a commutative ring. Then the set of sequences on R with the operations given above is a commutative ring.*

Proof. The proof follows from the facts about summation described above. For example, to prove associativity of multiplication, we must show that, for all natural numbers n ,

$$\bigoplus_{i=0}^n \left(\bigoplus_{k=0}^i a_k \cdot b_{i-k} \right) \cdot c_{n-i} \rightsquigarrow \bigoplus_{j=0}^n a_j \cdot \left(\bigoplus_{l=0}^{n-j} b_l \cdot c_{(n-j)-l} \right).$$

For this, we reason as follows

$$\bigoplus_{j=0}^n \bigoplus_{l=0}^{n-j} a_j \cdot (b_l \cdot c_{(n-j)-l}) \rightsquigarrow \bigoplus_{l=0}^n \bigoplus_{j=0}^l (a_l \cdot b_{k-l}) \cdot c_{n-l-(k-l)} \rightsquigarrow \bigoplus_{l=0}^n \bigoplus_{j=0}^l a_l \cdot (b_{k-l} \cdot c_{n-k}),$$

where the first path is given by Lemma 4.3 and associativity of multiplication in R . In the Coq proof this line of reasoning is put together with generous use of Lemma 4.1, (`funextfun`), several minor lemmas such as (`natsummationtimesdist1`), and associativity of R itself. \square

4.3 The apartness relation on formal power series

Although it is not used in the construction of the p -adic numbers, we mention here some results contained in the Coq files regarding apartness relations on formal power series.

Assume that R is a commutative ring with an apartness relation. Then there is an induced apartness relation on $R[[X]]$ given by setting (`fpsapart`)

$$a \# b \quad \text{if and only if} \quad \exists_{n:\mathbb{N}}. a_n \# b_n \tag{1}$$

for $a, b : R[[X]]$. This apartness relation is compatible with the ring operations and so we see that $R[[X]] : \mathbf{aCRng}$ (`acommrngfps`).

For R an apartness domain, provided that the apartness relation on R is decidable in the sense of Definition 3.3, it is possible to show that $R[[X]]$ is an apartness domain.

Proposition 4.5 (`apartdectoisaaintdomfps`). *For $R : \mathbf{aDom}$ with decidable apartness, the commutative ring $R[[X]]$ of formal power series is an apartness domain when equipped with the apartness relation (1).*

The proof of Proposition 4.5 is a consequence of the following lemma:

Lemma 4.6 (`leadingcoefficientapartdec`). *For $R : \mathbf{aDom}$ and $a : R[[X]]$, if $a_0 \# 0$, then for any $n : \mathbb{N}$ and $b : R[[X]]$, if $b_n \# 0$, then $(a \cdot b) \# 0$.*

Proof. The proof is by induction on n and is obvious in the base case. The induction case splits into two subcases depending on whether $b_0 \# 0$ or $b_0 \rightsquigarrow 0$. In the former case, $(a \cdot b)_0 \# 0$, whereas in the latter case the claim follows by applying the induction hypothesis to the sequence $b' : R[[X]]$ given by $b'_n := b_{n+1}$. \square

5 The Heyting field of fractions

The construction of the Heyting field of fractions from an apartness domain is a classical result in constructive algebra due to Heyting and we therefore give only a brief sketch of the details here.

Definition 5.1 (`aintdomzerosubmonoid`). Given $A : \mathbf{aDom}$, we denote by \tilde{A} the submonoid of A (with respect to the multiplicative structure of A) consisting of those $a : A$ such that $a \# 0$.

It follows (`commrngfrac`) that there exists a commutative ring $A[\tilde{A}^{-1}]$ obtained by localizing with respect to \tilde{A} . It remains to show there exists an apartness relation on $A[\tilde{A}^{-1}]$ which makes it into a Heyting field.

Definition 5.2 (`afldfracapartrel0`). For elements $a, c : A \times \tilde{A}$ we define

$$a \# c \quad \text{if and only if} \quad ((\pi_1 a) \cdot (\pi_2 c)) \# ((\pi_1 c) \cdot (\pi_2 a)).$$

This relation extends to a relation (`afldfracapartrel`) on $A[\tilde{A}^{-1}]$ and it is straightforward to show that it is an apartness relation (`afldfracapart`) which is compatible with the ring structure of $A[\tilde{A}^{-1}]$ (`afldfrac0`). For instance (`iscotransafldfracapartrelpre`), to see that it is cotransitive suppose given $(a, a') \# (c, c')$ and some (b, b') . Then, by the fact that A is an apartness domain, we see that $a \cdot c' \cdot b' \# c \cdot a' \cdot b'$. Therefore, by cotransitivity of the apartness relation of A , we have that either $a \cdot c' \cdot b' \# b \cdot a' \cdot c'$ or $b \cdot a' \cdot c' \# c \cdot a' \cdot b'$. In the former case it follows that $a \cdot b' \# b \cdot a'$. I.e., $(a, a') \# (b, b')$. In the latter case it similarly follows that $(b, b') \# (c, c')$.

Given $a \in A \times \tilde{A}$ such that $a \# 0$, we have $\pi_1(a) \# 0$ and therefore, we take a^{-1} to be given by the pair $(\pi_2(a), \pi_1(a))$. This definition extends to a definition of the inverse of an element apart from 0 in $A[\tilde{A}^{-1}]$ and it is straightforward to show that this gives makes $A[\tilde{A}^{-1}]$ a Heyting field:

Theorem 5.3 (`afldfracisafld`). *For $A : \mathbf{aDom}$, with the definitions given above, $A[\tilde{A}^{-1}]$ forms a Heyting field.*

We refer to the Heyting field from Theorem 5.3 as the **Heyting field of fractions** of A and we write $\mathbf{Frac}(A)$ for it.

6 The p -adic numbers

The p -adic numbers were invented about one hundred years ago by German mathematician K. Hensel.

6.1 Basic number theory

The following definition is the relation of integer divisibility, and is given as a two part definition in the Coq file. The first part says that, given three integers n, m, k , if the product of n and k is m , then n divides m . The general definition starts only with n and m , and appeals to the existence of k .

Definition 6.1 (`hzdiv0` and `hzdiv`). Let n and m be integers. We write $n|m$ for the type

$$n|m := \exists_{k:\mathbb{Z}}.(m \rightsquigarrow n \cdot k)$$

and we say that n **divides** m when $n|m$ is inhabited.

The division algorithm is then shown to hold via a series of steps. First, we prove the division algorithm for natural numbers. Recall that `pr1` and `pr2` are defined as projections onto the base and “specialization” to a fiber:

Lemma 6.2 (`dvalgorithmnonneg`). *For n and m of type `nat`, with m nonzero, there exists a term $qr : (\mathbb{Z} \times \mathbb{Z})$ such that there is a term of type*

$$n \rightsquigarrow (m \cdot \pi_1(qr)) + \pi_2(qr)$$

and there are proofs that $0 \leq \pi_2(qr) < m$.

The proof of Lemma 6.2 is by induction on n with, in the successor step, a case analysis on whether $(r' + 1) < m$ or $r' \rightsquigarrow m$ (that such a case analysis is possible follows from decidability of equality using `hzlehchoice` from the second author’s library). The proof of the general division algorithm is then done by a detailed case analysis (on whether n and m are negative, non-negative or propositionally equal to 0):

Theorem 6.3 (`dvalgorithmmexists`). *For n and m of type \mathbb{Z} with $m > 0$, the space of terms $qr : \mathbb{Z} \times \mathbb{Z}$ such that the types $n \rightsquigarrow (m \cdot \pi_1(qr)) + \pi_2(qr)$ and $0 \leq \pi_2(qr) < |m|$ are inhabited is contractible.*

Here, as throughout, *contractibility* corresponds to *unique existence* in the traditional setting. One consequence of the division algorithm is that we obtain the operations of taking the quotient and remainder of an integer modulo a non-negative integer (`hzquotientmod` and `hzremaindermod`). These two operations will play a role in a number of calculations in the sequel.

In addition to the division algorithm we also obtain the familiar Euclidean algorithm (again stated in terms of contractibility of an appropriate space):

Theorem 6.4 (`euclideanalgorithm`). *Let n and m be integers with $n \neq 0$. Then the space `hzgcd`(n, m) of greatest common divisors of n and m is contractible.*

We also obtain a form of the Bézout lemma:

Lemma 6.5 (`bezoutstrong`). *For all $m, n : \mathbb{Z}$ such that n is non-zero, the type of $ab : \mathbb{Z} \times \mathbb{Z}$ for which there exists a term of type $\gcd(n, m) \rightsquigarrow \pi_1(ab) \cdot n + \pi_2(ab) \cdot m$ is inhabited.*

Given $p : \mathbb{Z}$, the type of proofs that p is a prime is defined by setting

$$\text{isapriame}(p) := (1 < p) \times ((m|p) \rightarrow (m \rightsquigarrow 1) \vee (m \rightsquigarrow p)).$$

As a consequence of Lemma 6.5 we obtain

Theorem 6.6 (`acommrng_hzmod` and `ahzmod`). *For non-zero p of type \mathbb{Z} , $\mathbb{Z}/p\mathbb{Z}$ is a commutative ring with compatible apartness relation. When p is a prime, $\mathbb{Z}/p\mathbb{Z}$ is a Heyting field.*

Note that the apartness relation on $\mathbb{Z}/p\mathbb{Z}$ is the one induced by the fact that equality of $\mathbb{Z}/p\mathbb{Z}$ is decidable (`isdeceqhzmodp`).

6.2 The construction of \mathbb{Q}_p

Throughout this section we assume given a prime p . Explicitly, we require the proof witnessing the fact that p is a prime. We note though that for some of the results stated here it is only necessary that p be non-zero. We also introduce some notation for quotients and remainders modulo p . We denote by $\{a\}$ the quotient of a modulo p (`hzquotientmod`) and by $[a]$ the remainder of a modulo p .

We will now summarize our construction of the apartness domain \mathbb{Z}_p of p -adic integers.

Definition 6.7 (`precarry`). Given a formal power series a over \mathbb{Z} , we define a new formal power series $\mathbf{p}(a)$ over \mathbb{Z} inductively by

$$\begin{aligned} \mathbf{p}(a)_0 &:= a_0 \\ \mathbf{p}(a)_{n+1} &:= a_{n+1} + \{\mathbf{p}(a)_n\}. \end{aligned}$$

Definition 6.8 (`carry`). Given a formal power series a over \mathbb{Z} , we define a new formal power series a^\natural over \mathbb{Z} by

$$(a^\natural)_n := [\mathbf{p}(a)_n].$$

We call a^\natural the **carried power series of a** .

Example 6.9. The formal power series $a = (4, 1, 8, 0, \dots)$ is sent to $\mathbf{p}(a) = (4, 2, 8, 2, 0, \dots)$ and to $a^\natural = (1, 2, 2, 2, 0, \dots)$.

The operation of carrying (mod p) for power series induces an equivalence relation \sim (`carryequiv`) on $\mathbb{Z}[[X]]$ by setting

$$a \sim b \quad \text{if and only if} \quad a^\natural \rightsquigarrow b^\natural.$$

Observe that $X - p \sim 0$. Furthermore, for any $a \in \mathbb{Z}[[X]]$, if $a \sim 0$, then there exist integers λ_i such that $a_0 = -\lambda_0 p$ and $a_{n+1} = -\lambda_{n+1} p + \lambda_n$. Using these facts it follows that \sim is the equivalence relation corresponding to the ideal $(X - p)$ in $\mathbb{Z}[[X]]$. Ultimately, once the theory of ideals has been developed in the Univalent Foundations Library, \mathbb{Z}_p will be constructed as the quotient of $\mathbb{Z}[[X]]$ by this ideal. However, because quotients of rings are given in the second author's library in terms of congruences, we here describe \mathbb{Z}_p using the corresponding congruence \sim .

We will now describe the proof that this relation is a congruence with respect to the ring operations on $\mathbb{Z}[[X]]$.

Lemma 6.10 (quotientprecarryplus). *For formal power series a and b over \mathbb{Z} ,*

$$\{\mathbf{p}(a + b)_n\} \rightsquigarrow \{\mathbf{p}(a)_n\} + \{\mathbf{p}(b)_n\} + \{\mathbf{p}(a^\natural + b^\natural)_n\}$$

for $n : \mathbb{N}$.

Proof. The proof is by induction on n . In the base case it is trivial and in the induction case it is by the following calculation:

$$\begin{aligned} \{\mathbf{p}(a + b)_{n+1}\} &\rightsquigarrow \{\mathbf{p}(a)_{n+1} + \mathbf{p}(b)_{n+1} + \{\mathbf{p}(a^\natural + b^\natural)_n\}\} \\ &\rightsquigarrow \{\mathbf{p}(a)_{n+1}\} + \{\mathbf{p}(b)_{n+1}\} + \{\mathbf{p}(a^\natural + b^\natural)_n\} + \{a_{n+1}^\natural + b_{n+1}^\natural + [\mathbf{p}(a^\natural + b^\natural)_n]\} \\ &\rightsquigarrow \{\mathbf{p}(a)_{n+1}\} + \{\mathbf{p}(b)_{n+1}\} + \{\mathbf{p}(a^\natural + b^\natural)_{n+1}\} \end{aligned}$$

where the first path is by definition of precarry and the induction hypothesis, the second path is by the familiar decomposition of the quotient of a sum, and the final path is by definition and the fact that the quotient of a remainder is zero. \square

The following observation is a consequence of Lemma 6.10.

Lemma 6.11 (carryandplus). *For a and b formal power series over \mathbb{Z} , $(a + b)^\natural \rightsquigarrow (a^\natural + b^\natural)^\natural$.*

Similarly, a straightforward induction gives us the following lemma:

Lemma 6.12 (precarryandtimes1). *Given formal power series a and b over \mathbb{Z} ,*

$$\{\mathbf{p}(a \cdot b)_n\} \rightsquigarrow (\{\mathbf{p}(a)\} \cdot b)_n + \{\mathbf{p}(a^\natural \cdot b)_n\}$$

for $n : \mathbb{N}$.

The proof that carrying is compatible with multiplication of power series is then an immediate consequence of Lemma 6.12:

Lemma 6.13 (carryandtimes). *Given formal power series a and b over \mathbb{Z} , $(a \cdot b)^\natural \rightsquigarrow (a^\natural \cdot b^\natural)^\natural$.*

It follows from Lemmas 6.11 and 6.13 that the quotient of $\mathbb{Z}[[X]]$ by the equivalence relation \sim is itself a commutative ring (`commrngofpadicints`). Indeed, it is the commutative ring \mathbb{Z}_p of *p-adic integers*. Moreover, there is an apartness relation (`padicapart`) on *p*-adic integers obtained as the extension of the relation (`padicapart0`)

$$a \# b \quad \text{if and only if} \quad \exists_{n:\mathbb{N}}. \neg(a_n^\natural \rightsquigarrow b_n^\natural), \quad (2)$$

for $a, b : \mathbb{Z}[[X]]$, to the *p*-adic integers. This apartness relation is straightforwardly seen to be compatible with the ring structure of \mathbb{Z}_p (`acommrngofpadicints`).

Theorem 6.14 (`padicintsareintdom,padicintegers`). *The commutative ring \mathbb{Z}_p with the apartness relation described above forms an apartness domain.*

Proof. It suffices to prove that for $a, b : \mathbb{Z}[[X]]$ such that $a \# 0$ and $b \# 0$ it follows that $a \cdot b \# 0$, where we are considering only the apartness relation (2). Since \mathbb{Z} has decidable equality, it follows (`leastelementprinciple`) that there are natural numbers k and m which are the least natural numbers such that $\neg(a_k^\natural \rightsquigarrow 0)$ and $\neg(b_m^\natural \rightsquigarrow 0)$, respectively. It then follows that $\neg((a \cdot b)_{k+m}^\natural \rightsquigarrow 0)$.

To see this, assume for a contradiction that there is a path $(a \cdot b)_{k+m}^\natural \rightsquigarrow 0$ and consider first the case where $k + m = 0$. Then we have that $a_0 \cdot b_0$ is congruent to 0 modulo p and therefore, since p is prime, either a_0 is congruent to 0 modulo p or b_0 is congruent to 0 modulo p . In either case we have obtained a contradiction.

On the other hand, when $k + m$ is a successor $k + m = n + 1$, we have that

$$(a \cdot b)_{k+m}^\natural \rightsquigarrow [(a^\natural \cdot b^\natural)_{k+m} + \{\mathbf{p}(a^\natural \cdot b^\natural)_n^\natural\}]. \quad (3)$$

By the choice of k and m it follows that there is a further term (`precarryandzeromult`) of type $\mathbf{p}(a^\natural \cdot b^\natural)_n^\natural \rightsquigarrow 0$. Therefore, we obtain a term of type

$$0 \rightsquigarrow [(a^\natural \cdot b^\natural)_{k+m}].$$

However, it is easy (`hzfpstimeswhenzero`) to see that $(a^\natural \cdot b^\natural)_{k+m} \rightsquigarrow a_k^\natural \cdot b_m^\natural$. So, since p is prime, either $a_k^\natural \rightsquigarrow 0$ or $b_m^\natural \rightsquigarrow 0$ is inhabited. In either case we obtain a contradiction. \square

Using Theorem 6.14, we now arrive at our definition of the *p*-adic numbers:

Definition 6.15 (`padics`). The Heyting field \mathbb{Q}_p of *p*-adic numbers is defined as the Heyting field of fractions of \mathbb{Z}_p :

$$\mathbb{Q}_p := \mathbf{Frac}(\mathbb{Z}_p).$$

7 Future directions: towards p -adic integrable systems

Next we present an outline of the work on p -adic integrable systems that we plan to carry out following this paper. The long term goal is to develop an analogue of the symplectic theory of finite-dimensional real integrable systems in [13, 14] for p -adic integrable systems in the univalent setting, and implement it in Coq.

We are beginning to explore this, and what we give next is a brief and informal glimpse of our plans. At this point this section is a discussion without rigorous descriptions as we are not yet convinced of the optimal definition of p -adic integrable system. We hope to convey the fact that the p -adic and real theories are expected to be different, and draw attention to the topic; in fact, we are not aware of a uniform treatment of p -adic integrable systems in the symplectic setting.

7.1 Definition of p -adic integrable systems

A word on the contrast between p -adic and real notions

We refer to [8, Section 3] for basic algebraic and topological aspects concerning the p -adic numbers. Many aspects do not match the intuition we have for the real numbers. For instance, there are no nontrivial connected sets and there are non-empty sets which are both compact and open. Other aspects are more familiar: on \mathbb{Q}_p there is an absolute value $|\cdot|$ and \mathbb{Q}_p is complete with respect to it, and there is an inclusion $\mathbb{Q} \rightarrow \mathbb{Q}_p$ with dense image. Continuity and differentiability of functions is defined in the usual way [8, Definitions 4.2.1, 4.2.2]. Continuous functions are uniformly continuous on compact sets, as in the real case.

The notions of continuity and differentiability extend to functions $f: U \subset (\mathbb{Q}_p)^n \rightarrow \mathbb{Q}_p$ of several variables (x_1, \dots, x_n) on open sets U of the Cartesian product $(\mathbb{Q}_p)^n$, in direct analogy with the real case, and in particular we have analogous definitions for partial derivatives $\frac{\partial f}{\partial x_i}$, for all $i = 1, \dots, n$. But although the definitions are the same, differentiability behaves differently in the p -adic case than in the real case. For instance, there are functions $f: \mathbb{Q}_p \rightarrow \mathbb{Q}_p$ which have zero derivative everywhere but are *not* locally constant. Also, the natural extension of the real mean value theorem to the p -adic case is false in general (although a version holds for sufficiently close points), as seen for instance by considering $f(x) = x^p - x$ between the extreme points $a = 0$ and $b = 1$. In this case, [8, Proposition 4.2.3] $f'(x) = px^{p-1} - 1$ and $f(a) = f(b) = 0$ and it is easy to check that any element “in between” a and b , that is, of the form $at + b(1 - t) = 1 - t$ for some t with $|t| \leq 1$, gives rise to a unit $f'(1 - t)$ in \mathbb{Z}_p .

These differences are an indication that the theory of p -adic integrable systems is not expected to be a direct extension of the theory of real integrable systems, even if the basic definitions are analogous. One can explore such theory classically only, but we hope to do it in the univalent setting, building on the constructions of \mathbb{Q}_p which we have given in the previous sections.

Integrable systems

We are here going to propose a notion of p -adic integrable systems in parallel with the commonly accepted notion of real integrable systems, at least in symplectic geometry.

Because in the univalent foundations, and in Coq, it is nontrivial to define manifolds, for now we are going to work with the p -adic Cartesian product

$$M := (\mathbb{Q}_p)^{2n} = \mathbb{Q}_p \times \dots (2n \text{ times}) \dots \times \mathbb{Q}_p$$

with coordinates $(x_1, y_1, \dots, x_n, y_n)$. In this way, we also avoid a discussion of differential or symplectic forms. Fix a p -adic measure on \mathbb{Q}_p , and endow M with the induced product measure.

On M we may consider differentiable functions in the p -adic sense⁴. The following is the formal extension of the definition of real integrable system in finite dimensions. There is, however, a critical point which is not clear to us at the moment, and that's why we restrict our definition to analytic maps, see Remark 7.2.

Definition 7.1. We will say that a (p -adic) analytic map $F := (f_1, \dots, f_n): M \rightarrow (\mathbb{Q}_p)^n$ is a **p -adic integrable system** if two conditions hold:

1. the collection f_1, \dots, f_n satisfies Hamilton's equations:

$$\sum_{k=1}^n \frac{\partial f_i}{\partial x_k} \frac{\partial f_j}{\partial y_k} - \frac{\partial f_i}{\partial y_k} \frac{\partial f_j}{\partial x_k} = 0, \quad \forall 1 \leq i \leq j \leq n. \quad (4)$$

2. the set where the n formal differentials

$$dp_i := \left(\frac{\partial f_i}{\partial x_1}, \dots, \frac{\partial f_i}{\partial x_n}, \frac{\partial f_i}{\partial y_1}, \dots, \frac{\partial f_i}{\partial y_n} \right), \quad \forall 1 \leq i \leq n$$

are linearly dependent has p -adic measure 0.

That is, there exists a p -adic measure 0 set A such that df_1, \dots, df_n are linearly independent on $M \setminus A$. The points where df_1, \dots, df_n are linearly dependent are called *singularities*.

Remark 7.2. This remark explains why we have to restrict to analytic functions in Definition 7.1, when in the real theory one likes to include all smooth functions in the definition of integrable system. There are many interesting, nontrivial p -adic functions that are smooth and have zero derivative everywhere. *However this is not possible if one restricts to analytic functions.* Therefore if f is a smooth solution to a linear differential equation, we could add to f any of these nontrivial functions with zero derivative and obtain a new solution. It follows that all collections of n smooth functions f_1, \dots, f_n which are smooth and have zero derivative everywhere would also form a kind of integrable system, but a very "degenerate" one (in the sense that the differentials df_1, \dots, df_n would not be linearly independent almost

⁴for now we are thinking only of polynomials on $2n$ -variables, which are easy to deal with in Coq.

everywhere as it is normally required for real integrable systems). So this undesirable case does not occur. However, adding functions with zero derivative to an existing system would be unavoidable, giving rise to a new, seemingly very different, p -adic integrable system. We currently understand neither what this means geometrically, nor what it implies for the development of the theory.

7.2 Future plans

The following is a rough outline of what we would like to do next.

Towards p -adic symplectic geometry

- ▷ *p -adic manifolds*: formalize the notion of p -adic manifold in the univalent Foundations with Coq. Formalize Serre’s theorem [17] classifying compact p -adic manifolds.
- ▷ *p -adic symplectic forms*: a p -adic symplectic form ω may be defined as in the real case. The closedness condition $d\omega = 0$ makes sense in the p -adic setting, and so does the non-degeneracy condition (in fact, over any field). In the real setting, a theorem of Darboux says that all symplectic forms are locally equivalent, so real symplectic manifolds have no local invariants. It is natural to wonder whether this result holds in the p -adic setting “as is”. Because of our previous discussion (see Remark 7.2) one should probably restrict to the analytic setting since $d\omega = 0$ is in fact a system of partial differential equations. Darboux’s theorem plays a leading role in the theory of real integrable systems.

Towards p -adic integrable systems: basic theory

- ▷ *construction of p -adic integrable systems*: define p -adic integrable systems on p -adic manifolds, not just $(\mathbb{Q}_p)^n$, and implement this in the univalent foundations using Coq.
- ▷ *p -adic local and semiglobal theory*: develop the local and semilocal theory of p -adic integrable systems in Coq. The local theory basically refers to local models, and the semilocal theory refers to local models in neighborhoods of fibers. One is interested in both the topological and symplectic classification of such models. We are not aware of results describing the topological, or symplectic, structure of regular or singular fibers in the p -adic setting.

In the real case, the regular fibers and their neighborhoods are understood (this is the famous Action-Angle Theorem due to Mineur and Arnold.) The singular fibers may be complicated and not are yet well understood in the real setting either (if one restrict to the real analytic setting, then the theory is better understood).

Towards p -adic toric and semitoric systems

- ▷ *p -adic toric systems*: a particular class of real integrable systems which has been thoroughly studied and is well understood, is that of toric integrable systems $F = (f_1, \dots, f_n)$ on $2n$ -dimensional compact symplectic manifolds (M, ω) . These are systems in which each component f_i generates a flow which is periodic of a fixed period. In this case, F is called a *momentum map*. Atiyah [1], Guillemin-Sternberg [9] and Delzant [7] proved a series of striking theorems concerning these systems in the 1980s, which in particular led to complete combinatorial classification in terms of convex polytopes by Delzant (these convex polytopes are nothing by the images of M under F). A theorem of Serre [17] classifies compact p -adic analytic varieties. If on these varieties we would consider actions of the p -adic n -torus, we do not know to what extent the above results could be extended. If in Definition 7.1 one allows smooth non-analytic functions, these results would not hold (see Remark 7.2).
- ▷ *p -adic semitoric systems*: give a classification of p -adic integrable systems under some periodicity condition in analogy with [13, 14].

Spectral questions for p -adic integrable systems

Here we restrict to the systems in the previous section, for which we know that in the real case a full classification may be given.

- ▷ *Inverse spectral problems*: construct algorithms to solve inverse spectral problems about quantum integrable systems. The leading question in the real case is: given the spectrum, can one recover the system from it?
- ▷ *Numerical implementation of inverse spectral problems*: constructing numerically accurate algorithms to solve inverse spectral problems.

The first subsection above should be within reach. We expect the second and third subsections to be substantial. The fourth one depends on the third and it is difficult to predict how complicated it will be.

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Appendix: Coq code

Disclaimer: The libraries summarized and listed below are in preliminary form and are actively being improved and extended by the authors and others. As such, we advise interested readers to consult also with the most recent versions, which need not agree in form and content with the libraries described here.

The Coq code is included in full below for easy reference by the reader. We also expect to make it available on the webpages of the first and third authors. For easy reference, we include here a brief sketch of the contents of each of the files. *It is worth remarking that all of the files described here rely upon the second author's Coq library.* For more on this library we refer the reader to the library itself and to the tutorial [15]. For quick reference, Figures 1 and 2 give the dependences of the second author's library and the library associated with this paper, respectively.

Of the new files, the file `lemmas.v` contains a number of small lemmas which, such as basic facts about apartness relations, some lemmas on rings, *et cetera*, which are required by the other files. The file `fps.v` contains all of the material on formal power series. The construction of the Heyting field of fractions can be found in `frac.v`. The basic number theoretic results which we require are in `zmodp.v`. Finally, the construction of the p -adic numbers is given in `padics.v`.

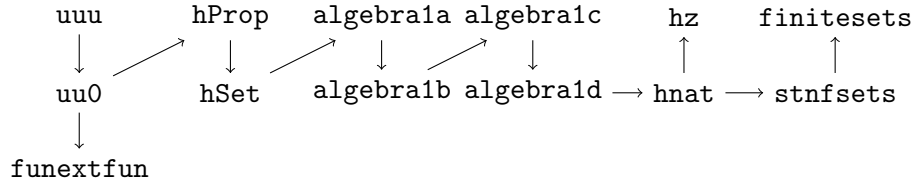


Figure 1: Dependence diagram of the second author's Coq library

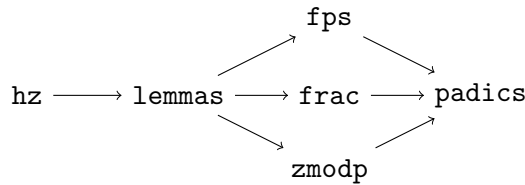


Figure 2: Dependence diagram of the additional Coq files for the p -adics

7.3 The file lemmas.v

```
(** *Fixing notation, terminology and basic lemmas *)

(** By Alvaro Pelayo, Vladimir Voevodsky and Michael A. Warren *)

(** Settings *)

Add Rec LoadPath "../Generalities". Add Rec LoadPath "../hlevel1".
Add Rec LoadPath "../hlevel2".

Unset Automatic Introduction. (** This line has to be removed for the
file to compile with Coq8.2 *)

(** Imports *)

Require Export hSet.

Require Export hnat.

Require Export hz.

(*Require Export finitesets.

Require Export stnfsets.*)

(** Fixing some notation *)

(** * Notation, terminology and very basic facts *)

Notation "x ~> y" := ( paths x y ) ( at level 50 ) : type_scope.

Implicit Arguments tpair [ T P ].

Lemma pathintotalfiber ( B : UU ) ( E : B -> UU ) ( b0 b1 : B ) ( e0 :
E b0 ) ( e1 : E b1 ) ( p0 : b0 ~> b1 ) ( p1 : transportf E p0 e0 ~> e1
) : ( tpair b0 e0 ) ~> ( tpair b1 e1 ). Proof. intros. destruct p0,
p1. apply idpath. Defined.

Definition neq ( X : UU0 ) : X -> X -> hProp := fun x y : X =>
hProppair (neg (x ~> y)) (isapropneg (x ~> y)).

Definition pathintotalpr1 { B : UU } { E : B -> UU } { v w : total2 E
} ( p : v ~> w ) : ( pr1 v ) ~> ( pr1 w ) := maponpaths ( fun x => pr1
x ) p.

Lemma isinclinjinj { A B : UU } { f : A -> B } ( p : isincl f ) { a b :
A } ( p : f a ~> f b ) : a ~> b. Proof. intros. set ( q := p ( f a )
). set ( a' := hfiberpair f a ( idpath ( f a ) ) ). set ( b' :=
hfiberpair f b ( pathsinv0 p0 ) ). assert ( a' ~> b' ) as p1. apply (
p ( f a ) ). apply ( pathintotalpr1 p1 ). Defined.

(** * I. Lemmas on natural numbers *)

Lemma minus0r ( n : nat ) : minus n 0 ~> n. Proof. intros
n. destruct n. apply idpath. apply idpath. Defined.

Lemma minusnn0 ( n : nat ) : minus n n ~> 0%nat. Proof.
intro. induction n. apply idpath. assumption. Defined.

Lemma minusn1 ( n : nat ) : minus ( S n ) 1 ~> n. Proof.
intro. destruct n. apply idpath. apply idpath. Defined.

Lemma minusnlnon0 ( n : nat ) ( p : natlth 0 n ) : S ( minus n 1 ) ~>
```

```
n. Proof. intro. destruct n. intro p. assert empty. exact (
isirreflnatlth 0%nat p ). contradiction. intro. apply
maponpaths. apply minus0r. Defined.
```

```
Lemma minusleh ( n m : nat ) : natleh ( minus n m ) n. Proof. intros
n. induction n. intros m. apply isreflnatleh. intros m. destruct
m. apply isreflnatleh. exact ( istransnatleh ( minus n m ) n ( S n ) (
IHn m ) ( natlthtoleh n ( S n ) ( natlthnsn n ) ) ). Defined.
```

```
Lemma minusileh { n m : nat } ( p : natlth 0 n ) ( q : natlth 0 m ) (
r : natleh n m ) : natleh ( minus n 1 ) ( minus m 1 ). Proof. intro
n. destruct n. auto. intros m p q r. destruct m. assert empty. exact (
isirreflnatlth 0%nat q ). contradiction. assert ( natleh n m ) as
a. apply r. assert ( natleh ( minus n 0%nat ) m ) as a0. exact (
transportf ( fun x : nat => natleh x m ) ( pathsinv0 ( minus0r n ) ) a
). exact ( transportf ( fun x : nat => natleh ( minus n 0 ) x ) (
pathsinv0 ( minus0r m ) ) a0 ). Defined.
```

```
Lemma minuslth ( n m : nat ) ( p : natlth 0 n ) ( q : natlth 0 m ) :
natlth ( minus n m ) n. Proof. intro n. destruct n. auto. intros m p
q. destruct m. assert empty. exact ( isirreflnatlth 0%nat q
). contradiction. apply ( natlehlthtrans _ n _ ). apply ( minusleh n m
). apply natlthnsn. Defined.
```

```
Lemma natlthstoleh ( n m : nat ) : natlth m ( S n ) -> natleh m n.
Proof. intro. induction n. intros m p. destruct m. apply
isreflnatleh. assert ( natlth m 0 ) as q. apply p. intro. unfold
natlth in q. exact ( negnatgth0n m q ). intros m p. destruct m. apply
natleh0n. apply ( IHn m ). assumption. Defined.
```

```
Lemma natlthminus0 { n m : nat } ( p : natlth m n ) : natlth 0 ( minus
n m ). Proof. intro n. induction n. intros m p. assert empty. exact
( negnatlthn0 m p ). contradiction. intros m p. destruct
m. auto. apply IHn. apply p. Defined.
```

```
Lemma natlthsnminussmn ( n m : nat ) ( p : natlth m n ) : natlth (
minus ( S n ) ( S m ) ) ( S n ). Proof. intro. induction n. intros m
p. assert empty. apply
nopathsfalsestottrue. assumption. contradiction. intros m p. destruct
m. assert ( minus ( S ( S n ) ) 1 ~> S n ) as f. destruct
n. auto. auto. rewrite f. apply natlthnsn. apply ( istransnatlth _ ( S
n ) _ ). apply IHn. assumption. apply natlthnsn. Defined.
```

```
Lemma natlehsnminussmn ( n m : nat ) ( p : natleh m n ) : natleh (
minus ( S n ) ( S m ) ) ( S n ). Proof. intro n. induction n. intros
m p X. assert empty. apply nopathsfalsestottrue. assumption.
assumption. intros m p. destruct m. apply natlthtoleh. apply
natlthnsn. apply ( istransnatleh _ ( S n ) _ ). apply
IHn. assumption. apply natlthtoleh. apply natlthnsn. Defined.
```

```
Lemma pathssminus ( n m : nat ) ( p : natlth m ( S n ) ) : S ( minus n
m ) ~> minus ( S n ) m. Proof. intro n. induction n. intros m
p. destruct m. auto. assert empty. apply nopathstruetofalse. apply
pathsinv0. assumption. contradiction. intros m p. destruct
m. auto. apply IHn. apply p. Defined.
```

```
Lemma natlehsminus ( n m : nat ) : natleh ( minus ( S n ) m ) ( S (
minus n m ) ). Proof. intro n. induction n. intros m X. apply
nopathstruetofalse. apply pathsinv0. destruct
m. assumption. assumption. intros m. destruct m. apply
isreflnatleh. apply IHn. Defined.
```

Lemma natlthssminus {n m l : nat} (p : natlth m (S n)) (q : natlth l (S (minus (S n) m))) : natlth l (S (S n)). Proof. intro n. intros m l p q. apply (natlthlehtans _ (S (minus (S n) m))). assumption. destruct m. apply isreflnatleh. apply natlthtoleh. apply natlthsnminussmn. assumption. Defined.

Lemma natdoubleminus {n k l : nat} (p : natleh k n) (q : natleh l k) : (minus n k) ~> (minus (minus n l) (minus k l)). Proof. intro n. induction n. auto. intros k l p q. destruct k. destruct l. auto. assert empty. exact (negnatlehsn0 l q). contradiction. destruct l. auto. apply (IHn k l). assumption. assumption. Defined.

Lemma minusnlehl {n m : nat} (p : natlth m n) : natleh m (minus n 1). Proof. intro n. destruct n. intros m p. assert empty. exact (negnatlthn0 m p). contradiction. intros m p. destruct m. apply natleh0n. apply natlthsnoleh. change (minus (S n) 1) with (minus n 0). rewrite minus0r. assumption. Defined.

Lemma doubleminuslehtpaths {n m : nat} (p : natleh m n) : minus n (minus n m) ~> m. Proof. intro n. induction n. intros m p. destruct (natlehchoice m 0 p) as [h | k]. assert empty. apply negnatlthn0 with (n := m). assumption. contradiction. simpl. apply pathsinv0. assumption.

intros. destruct m. simpl. apply minusnn0. change (minus (S n) (minus n m) ~> S m). rewrite <- pathssminus. rewrite IHn. apply idpath. assumption. apply (minuslth (S n) (S m)). apply (natlehlthtrans _ n). apply natleh0n. apply natlthsn. apply (natlehlthtrans _ m). apply natleh0n. apply natlthsn. Defined.

Lemma boolnegtrueimplfalse {v : bool} (p : neg (v ~> true)) : v ~> false. Proof. intros. destruct v. assert empty. apply p. auto. contradiction. auto. Defined.

Definition natcoface {i : nat} : nat -> nat. Proof. intros i n. destruct (natgtb i n). exact n. exact (S n). Defined.

Lemma natcofaceleh {i n upper : nat} (p : natleh n upper) : natleh (natcoface i n) (S upper). Proof. intros. unfold natcoface. destruct (natgtb i n). apply natlthtoleh. apply (natlehlthtrans _ upper). assumption. apply natlthsn. apply p. Defined.

Lemma natgehimplnatgtbfalse {m n : nat} (p : natgeh m n) : natgtb m n ~> false. Proof. intros. unfold natgeh in p. unfold natgtb in p. apply boolnegtrueimplfalse. intro q. apply p. auto. Defined.

Definition natcofacere retract {i : nat} : nat -> nat. Proof. intros i n. destruct (natgtb i n). exact n. exact (minus n 1). Defined.

Lemma natcofacere tractisretract {i : nat} : funcomp (natcoface i) (natcofacere retract i) ~> idfun nat. Proof. intro i. apply funextfun. intro n. unfold funcomp. set (c := natlthorgeh n i). destruct c as [h | k]. unfold natcoface. rewrite h. unfold natcofacere retract. rewrite h. apply idpath. assert (natgtb i n ~> false) as f. apply natgehimplnatgtbfalse. assumption. unfold natcoface. rewrite f. unfold natcofacere retract. assert (natgtb i (S n) ~> false) as ff. apply natgehimplnatgtbfalse. apply (istransnatgeh _ n). apply natgthtogeh. apply natgthsn. assumption. rewrite ff. rewrite minusn1. apply idpath. Defined.

Lemma isinjnatcoface {i x y : nat} : natcoface i x ~> natcoface i y -> x ~> y. Proof. intros i x y p. change x with ((idfun _) x). rewrite <- (natcofacere tractisretract i). change y with ((idfun _) y). rewrite <- (natcofacere tractisretract i). unfold funcomp. rewrite p. apply idpath. Defined.

Lemma natlehdecomp {b a : nat} : hexists (fun c : nat => (a + c)%nat ~> b) -> natleh a b. Proof. intro b. induction b. intros a p. apply p. intro t. destruct t as [c f]. destruct a. apply isreflnatleh. assert empty. simpl in f. exact (negpathssx0 (a + c) f). contradiction. intros a p. apply p. intro t. destruct t as [c f]. destruct a. apply natleh0n. assert (natleh a b) as q. simpl in f. apply IHb. intro P. intro s. apply s. split with c. apply invmaponpathsS. assumption. apply q. Defined.

Lemma natdivleh {a b k : nat} (f : (a * k)%nat ~> b) : coprod (natleh a b) (b ~> 0%nat). Proof. intros. destruct k. rewrite natmultcomm in f. simpl in f. apply ii2. apply pathsinv0. assumption. rewrite natmultcomm in f. simpl in f. apply ii1. apply natlehdecomp. intro P. intro g. apply g. split with (k * a)%nat. assumption. Defined.

(** * II. Lemmas on rings *)

Open Scope rng_scope.

Lemma rngminusdistr {X : commrng} (a b c : X) : a * (b - c) ~> (a * b - a * c). Proof. intros. rewrite rngldistr. rewrite rngmultminus. apply idpath. Defined.

Lemma rngminusdistl {X : commrng} (a b c : X) : (b - c) * a ~> (b * a - c * a). Proof. intros. rewrite rngrdistr. rewrite rnglmultminus. apply idpath. Defined.

Lemma multinvmultstable (A : commrng) (a b : A) (p : multinvpair A a) (q : multinvpair A b) : multinvpair A (a * b). Proof. intros. destruct p as [a' p]. destruct q as [b' q]. split with (b' * a'). split. change ((b' * a') * (a * b))%rng ~> (@rngunel2 A)). rewrite (rngassoc2 A b'). rewrite <- (rngassoc2 A a'). change (dirprod ((a' * a)%rng ~> (@rngunel2 A)) ((a * a')%rng ~> (@rngunel2 A))) in p. change (dirprod ((b' * b)%rng ~> (@rngunel2 A)) ((b * b')%rng ~> (@rngunel2 A))) in q. rewrite <- (pr1 q). apply maponpaths. assert (a' * a * b ~> 1 * b) as f. apply (maponpaths (fun x => x * b) (pr1 p)). rewrite rnglunax2 in f. assumption. change ((a * b) * (b' * a'))%rng ~> (@rngunel2 A)). rewrite (rngassoc2 A a). rewrite <- (rngassoc2 A b). change (dirprod ((a' * a)%rng ~> (@rngunel2 A)) ((a * a')%rng ~> (@rngunel2 A))) in p. change (dirprod ((b' * b)%rng ~> (@rngunel2 A)) ((b * b')%rng ~> (@rngunel2 A))) in q. rewrite <- (pr2 q). rewrite (pr2 q). rewrite rnglunax2. apply p. Defined.

Lemma commrngaddinunique (X : commrng) (a b c : X) (p : @op1 X a b ~> @rngunel1 X) (q : @op1 X a c ~> @rngunel1 X) : b ~> c. Proof. intros. rewrite (pathsinv0 (rngrunax1 X b)). rewrite (pathsinv0 q). rewrite (pathsinv0 (rngassoc1 X _ _)). rewrite (rngcomm1 X b _). rewrite p. rewrite rnglunax1. apply idpath. Defined.

Lemma isapropmultinvpair (X : commrng) (a : X) : isaprop (multinvpair X a). Proof. intros. unfold isaprop. intros b c.

assert (b ~> c) as f. destruct b as [b b']. destruct c as [c c']. assert (b ~> c) as f0. rewrite <- (@rngunax2 X b). change (b * (@rngunel2 X)) with (b * 1)%multmonoid. rewrite <- (pr2 c'). change (b * (a * c))%rng ~> c). rewrite <- (rngassoc2 X). change (b * a)%rng with (b * a)%multmonoid. rewrite (pr1 b'). change ((@rngunel2 X) * c ~> c)%rng. apply rnglunax2. apply pathintotalfiber with (p0 := f0). assert (isaprop (dirprod (c * a ~> (@rngunel2 X)) (a * c ~> (@rngunel2 X)))) as is. apply isofhleveldirprod. apply (setproperty X). apply (setproperty X). apply is. split with f. intros g. assert (isaset (multinvpair

```

X a)) as is. unfold multinvpair. unfold invpair. change isaset
with (isofhlevel 2). apply isofhleveltotal2. apply (pr1 (pr1 (
  rigmultmonoid X))) . intros. apply isofhleveldirprod. apply
hlevelIntosn. apply (setproperty X). apply hlevelIntosn. apply (
  setproperty X). apply is. Defined.

```

Close Scope rng_scope.

(** * III. Lemmas on hz *)

Open Scope hz_scope.

```

Lemma hzaddinvplus (n m : hz) : - (n + m) ~> ( (- n) + (- m) )
). Proof. intros. apply commrngaddinvuniqu with (a := n + m).
apply rngrinvasi. assert ((n + m) + (- n + - m) ~> (n + - n + m
+ - m)) as i. assert (n + m + (- n + - m) ~> (n + (m + (- n +
- m)))) as i0. apply rngassoci. assert (n + (m + (- n + - m)
)) ~> (n + (m + - n + - m)) as i1. apply maponpaths. apply
pathsinv0. apply rngassoci. assert (n + (m + - n + - m) ~> (n + (
- n + m + - m))) as i2. apply maponpaths. apply (maponpaths (fun
x : _ => x + - m)). apply rngcomm1. assert (n + (- n + m + - m)
~> (n + (- n + m) + - m)) as i3. apply pathsinv0. apply
rngassoci. assert (n + (- n + m) + - m ~> (n + - n + m + - m))
as i4. apply pathsinv0. apply (maponpaths (fun x : _ => x + - m))
). apply rngassoci. exact (pathscomp0 i0 (pathscomp0 i1 (
pathscomp0 i2 (pathscomp0 i3 i4))))). assert (n + - n + m + - m
~> 0) as j. assert (n + - n + m + - m ~> (0 + m + - m)) as j0.
apply (maponpaths (fun x : _ => x + m + - m)). apply rngrinvasi.
assert (0 + m + - m ~> (m + - m)) as j1. apply (maponpaths (fun
x : _ => x + - m)). apply rnglunaxi. assert (m + - m ~> 0) as
j2. apply rngrinvasi. exact (pathscomp0 j0 (pathscomp0 j1 j2)).
exact (pathscomp0 i j). Defined.

```

```

Lemma hzgthsntogeh (n m : hz) (p : hzgth (n + 1) m) : hzgeh n m.
Proof. intros. set (c := hzgthchoice2 (n + 1) m). destruct c as
[h | k]. exact p. assert (hzgth n m) as a. exact (
hzgthandplusrinv n m 1 h). apply hzgthtogh. exact a. rewrite (
hzplusrcan n m 1 k). apply isreflhzgeh. Defined.

```

```

Lemma hzlthstoleh (n m : hz) (p : hzlth m (n + 1)) : hzleh n m.
Proof. intros. unfold hzlth in p. assert (hzgeh n m) as a. apply
hzgthsntogeh. exact p. exact a. Defined.

```

```

Lemma hzabsvalchoice (n : hz) : coprod (0%nat ~> (hzabsval n)) (
total2 (fun x : nat => S x ~> (hzabsval n))). Proof.
intros. destruct (natlehchoice _ _ (natleh0n (hzabsval n))) as [
l | r]. apply ii2. split with (minus (hzabsval n) 1). rewrite
pathssminus. change (minus (hzabsval n) 0 ~> hzabsval n). rewrite
minus0r. apply idpath. assumption. apply iii. assumption. Defined.

```

```

Lemma hzlthminusswap (n m : hz) (p : hzlth n m) : hzlth (- m) (
- n). Proof. intros. rewrite <- (hzplusl0 (- m)). rewrite <- (
hzrminus n). change (hzlth (n + - n + - m) (- n)). rewrite
hzplusassoc. rewrite (hzpluscomm (- n)). rewrite <-
hzplusassoc. assert (- n ~> (0 + - n)) as f. apply
pathsinv0. apply hzplusl0. assert (hzlth (n + - m + - n) (0 + - n
)) as q. apply hzlthandplusr. rewrite <- (hzrminus m). change (m
- m) with (m + - m). apply hzlthandplusr. assumption. rewrite <- f
in q. assumption. Defined.

```

```

Lemma hzlthminusequiv (n m : hz) : dirprod ((hzlth n m) -> (
hzlth 0 (m - n))) ((hzlth 0 (m - n)) -> (hzlth n m)).
Proof. intros. rewrite <- (hzrminus n). change (n - n) with (n +
- n). change (m - n) with (m + - n). split. intro p. apply
hzlthandplusr. assumption. intro p. rewrite <- (hzplusr0 n

```

```

). rewrite <- (hzplusr0 m). rewrite <- (hzlminus n). rewrite <-
2! hzplusassoc. apply hzlthandplusr. assumption. Defined.

```

```

Lemma hzlthminus (n m k : hz) (p : hzlth n k) (q : hzlth m k) (
q' : hzleh 0 m) : hzlth (n - m) k. Proof. intros. destruct (
hzlehchoice 0 m q') as [l | r]. apply (istranshzlth _ n _).
assert (hzlth (n - m) (n + 0)) as i0. rewrite <- (hzrminus m
). change (m - m) with (m + - m). rewrite <- (hzplusassoc
). apply hzlthandplusr. assert (hzlth (n + 0) (n + m)) as
i00. apply hzlthandplusl. assumption. rewrite (hzplusr0) in
i00. assumption. rewrite hzplusr0 in
i0. assumption. assumption. rewrite <- r. change (n - 0) with (n +
- 0). rewrite hzminuszero. rewrite (hzplusr0 n). assumption.
Defined.

```

```

Lemma hzabsvalandminuspos (n m : hz) (p : hzleh 0 n) (q : hzleh 0
m) : nattohz (hzabsval (n - m)) ~> nattohz (hzabsval (m - n)
). Proof. intros. destruct (hzlthorgeh n m) as [l | r]. assert
(hzlth (n - m) 0) as a. change (n - m) with (n + - m).
rewrite <- (hzrminus m). change (m - m) with (m + - m). apply (
hzlthandplusr). assumption. assert (hzlth 0 (m - n)) as b.
change (m - n) with (m + - n). rewrite <- (hzrminus n). change (
n - n) with (n + - n). apply hzlthandplusr. assumption. rewrite (
hzabsvalnth0 a). rewrite hzabsvalgth0. change (n - m) with (n +
- m). rewrite hzaddinvplus. change (- - m) with (- - m
)%rng. rewrite (rngminusminus). rewrite hzpluscomm. apply
idpath. apply b. destruct (hzgehchoice n m r) as [h | k]. assert
(hzlth 0 (n - m)) as a. change (n - m) with (n + - m).
rewrite <- (hzrminus m). change (m - m) with (m + - m). apply
hzlthandplusr. assumption. assert (hzlth (m - n) 0) as b. change
(m - n) with (m + - n). rewrite <- (hzrminus n). apply
hzlthandplusr. apply h. rewrite (hzabsvalnth0 b). rewrite (
hzabsvalgth0). change ((n + - m) ~> - (m + - n)). rewrite
hzaddinvplus. change (- - n) with (- - n)%rng. rewrite
rngminusminus. rewrite hzpluscomm. apply idpath. apply a. rewrite
k. apply idpath. Defined.

```

```

Lemma hzabsvalneg0 (n : hz) (p : hzneq 0 n) : hzlth 0 (nattohz (
hzabsval n)). Proof. intros. destruct (hzneqchoice 0 n p) as [
left | right]. rewrite hzabsvalnth0. apply hzlth0andminus. apply
left. apply left. rewrite hzabsvalgth0. assumption. apply right.
Defined.

```

```

Definition hzrdistr (a b c : hz) : (a + b) * c ~> ((a * c) + (
b * c)) := rngrdistr hz a b c.

```

```

Definition hzltdistr (a b c : hz) : c * (a + b) ~> ((c * a) + (
c * b)) := rngldistr hz a b c.

```

```

Lemma hzabsvaland1 : hzabsval 1 ~> 1%nat. Proof. apply (isinclisinj
isinclnattohz). rewrite hzabsvalgth0. rewrite nattohzand1. apply
idpath. rewrite <- (hzplusl0 1). apply (hzlthnsn 0). Defined.

```

```

Lemma hzabsvalandplusnonneg (n m : hz) (p : hzleh 0 n) (q : hzleh
0 m) : hzabsval (n + m) ~> ((hzabsval n) + (hzabsval m))%nat.
Proof. intros. assert (hzleh 0 (n + m)) as r. rewrite <- (
hzrminus n). change (n - n) with (n + - n). apply
hzleh0andplusl. apply (istranshzleh _ 0 _). apply
hzgeh0andminus. apply p. assumption. apply (isinclisinj
isinclnattohz). rewrite nattohzandplus. rewrite 3!
hzabsvalgeh0. apply idpath. apply q. apply p. apply r. Defined.

```

```

Lemma hzabsvalandplusneg (n m : hz) (p : hzlth n 0) (q : hzlth m
0) : hzabsval (n + m) ~> ((hzabsval n) + (hzabsval m))%nat.
Proof. intros. assert (hzlth (n + m) 0) as r. rewrite <- (

```

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hzrminus n ). change ( n - n ) with ( n + - n ). apply
hzlthandplusl. apply ( istranshzlth _ 0 _ ). assumption. apply
hzlth0andminus. assumption. apply ( isinclisinj isinclnattohz ).
rewrite nattohzandplus. rewrite 3! hzabsvalth0. rewrite
hzaddinplus. apply idpath. apply q. apply p. apply r. Defined.

Lemma hzabsvalandnattohz ( n : nat ) : hzabsval ( nattohz n ) ~> n.
Proof. induction n. rewrite nattohzand0. rewrite hzabsval0. apply
idpath. rewrite nattohzandS. rewrite hzabsvalandplusnonneg. rewrite
hzabsvalandl. simpl. apply maponpaths. assumption. rewrite <- (
hzplusl0 1). apply hzlthtoleh. apply ( hzlthnsn 0 ). rewrite <-
nattohzand0. apply nattohzandleh. apply natleh0n. Defined.

Lemma hzabsvalandlth ( n m : hz ) ( p : hzleh 0 n ) ( p' : hzlth n m )
: natlth ( hzabsval n ) ( hzabsval m ). Proof. intros. destruct (
natlthorgeh ( hzabsval n ) ( hzabsval m ) ) as [ h | k ].
assumption. assert empty. apply ( isirreflhzlth m ). apply (
hzlehlthtrans _ n _ ). rewrite <- ( hzabsvalgeh0 ). rewrite <- (
hzabsvalgeh0 p ). apply nattohzandleh. apply k. apply
hzgthtogeh. apply ( hzgthgehtrans _ n _ ). apply p'. apply
p. assumption. contradiction. Defined.

Lemma nattohzandlthin ( n m : nat ) ( p : hzlth ( nattohz n )
(nattohz m ) ) : natlth n m. Proof. intros. rewrite <- (
hzabsvalandnattohz n ). rewrite <- ( hzabsvalandnattohz m ). apply
hzabsvalandlth. change 0 with ( nattohz 0%nat ). apply nattohzandleh.
apply natleh0n. assumption. Defined.

Close Scope hz_scope.

(** * IV. Generalities on apartness relations *)

Definition iscomparel { X : UU0 } ( R : hrel X ) := forall x y z : X,
R x y -> coprod ( R x z ) ( R z y ).

Definition isapart { X : UU0 } ( R : hrel X ) := dirprod ( isirrefl R
) ( dirprod ( issymm R ) ( iscotrans R ) ).

Definition istightapart { X : UU0 } ( R : hrel X ) := dirprod (
isapart R ) ( forall x y : X, neg ( R x y ) -> ( x ~> y ) ).

Definition apart ( X : hSet ) := total2 ( fun R : hrel X => isapart R
).

Definition isbinopapartl { X : hSet } ( R : apart X ) ( opp : binop X
) := forall a b c : X, ( ( pr1 R ) ( opp a b ) ( opp a c ) ) -> ( pr1
R ) b c.

Definition isbinopapart { X : hSet } ( R : apart X ) ( opp : binop X
) := forall a b c : X, ( pr1 R ) ( opp b a ) ( opp c a ) -> ( pr1 R )
b c.

Definition isbinopapart { X : hSet } ( R : apart X ) ( opp : binop X )
:= dirprod ( isbinopapartl R opp ) ( isbinopapart R opp ).

Lemma deceqtoneqapart { X : UU0 } ( is : isdeceq X ) : isapart ( neq X
). Proof. intros. split. intros a. intro p. apply p. apply idpath.
split. intros a b p q. apply p. apply pathsinv0. assumption. intros a
c b p P s. apply s. destruct ( is a c ) as [ l | r ]. apply
ii2. rewrite <- l. assumption. apply iii. assumption. Defined.

Definition isapartdec { X : hSet } ( R : apart X ) := forall a b : X,
coprod ( ( pr1 R ) a b ) ( a ~> b ).

Lemma isapartdectodeceq { X : hSet } ( R : apart X ) ( is : isapartdec
R ) : isdeceq X. Proof. intros X R is y z. destruct ( is y z ) as [

```

```

l | r ]. apply ii2. intros f. apply ( ( pr1 ( pr2 R ) ) z ). rewrite f
in l. assumption. apply ii1. assumption. Defined.

```

```

Lemma isdeceqtoisapartdec ( X : hSet ) ( is : isdeceq X ) : isapartdec
( tpair _ ( deceqtoneqapart is ) ). Proof. intros X is a b. destruct
( is a b ) as [ l | r ]. apply ii2. assumption. apply iii. intros
f. apply r. assumption. Defined.

```

```

(** * V. Apartness relations on rings *)

```

```

Open Scope rng_scope.

```

```

Definition acommrng := total2 ( fun X : commrng => total2 ( fun R :
apart X => dirprod ( isbinopapart R ( @op1 X ) ) ( isbinopapart R (
@op2 X ) ) ) ).

```

```

Definition acommrngpair := tpair ( P := fun X : commrng => total2 (
fun R : apart X => dirprod ( isbinopapart R ( @op1 X ) ) (
isbinopapart R ( @op2 X ) ) ) ). Definition acommrngconstr :=
accommrngpair.

```

```

Definition acommrngtocommrng : acommrng -> commrng := @pr1 _ _ .
Coercion acommrngtocommrng : acommrng ->> commrng.

```

```

Definition acommrngapartrel ( X : acommrng ) := pr1 ( pr1 ( pr2 X ) ).
Notation " a # b " := ( acommrngapartrel _ a b ) ( at level 50 ) :
rng_scope.

```

```

Definition acommrng_aadd ( X : acommrng ) : isbinopapart ( pr1 ( pr2 X
) ) op1 := ( pr1 ( pr2 ( pr2 X ) ) ). Definition acommrng_amult ( X :
accommrng ) : isbinopapart ( pr1 ( pr2 X ) ) op2 := ( pr2 ( pr2 ( pr2 X
) ) ). Definition acommrng_airsrefl ( X : acommrng ) : isirrefl ( pr1
( pr1 ( pr2 X ) ) ) := pr1 ( pr2 ( pr1 ( pr2 X ) ) ). Definition
accommrng_asyymm ( X : acommrng ) : issymm ( pr1 ( pr1 ( pr2 X ) ) ) :=
pr1 ( pr2 ( pr2 ( pr1 ( pr2 X ) ) ) ). Definition acommrng_acotrans (
X : acommrng ) : iscotrans ( pr1 ( pr1 ( pr2 X ) ) ) := pr2 ( pr2 (
pr2 ( pr1 ( pr2 X ) ) ) ).

```

```

Definition aintdom := total2 ( fun A : acommrng => dirprod ( (
rngunel2 ( X := A ) ) # 0 ) ( forall a b : A, ( a # 0 ) -> ( b # 0 )
-> ( ( a * b ) # 0 ) ) ).

```

```

Definition aintdompair := tpair ( P := fun A : acommrng => dirprod ( (
rngunel2 ( X := A ) ) # 0 ) ( forall a b : A, ( a # 0 ) -> ( b # 0 )
-> ( ( a * b ) # 0 ) ) ). Definition aintdomconstr := aintdompair.

```

```

Definition pr1aintdom : aintdom -> acommrng := @pr1 _ _ . Coercion
pr1aintdom : aintdom ->> acommrng.

```

```

Definition aintdomazerosubmonoid ( A : aintdom ) : @subabmonoids (
rngmultabmonoid A ). Proof. intros. split with ( fun x : A => ( x # 0
) ). split. intros a b. simpl in *. apply A. apply a. apply b. apply
A. Defined.

```

```

Definition isaafld ( A : acommrng ) := dirprod ( ( rngunel2 ( X := A
) ) # 0 ) ( forall x : A, x # 0 -> multinvpair A x ).

```

```

Definition aflld := total2 ( fun A : acommrng => isaafld A ).
Definition aflldpair ( A : acommrng ) ( is : isaafld A ) : aflld :=
tpair A is . Definition priaflld : aflld -> acommrng := @pr1 _ _ .
Coercion priaflld : aflld ->> acommrng.

```

```

Lemma aflldinvertibletoazero ( A : aflld ) ( a : A ) ( p : multinvpair A
a ) : a # 0. Proof. intros. destruct p as [ a' p ]. assert ( a' * a
# 0 ) as q. change ( a' * a # 0 ). assert ( a' * a ~> a * a' ) as
f. apply ( rngcomm2 A ). assert ( a * a' ~> 1 ) as g. apply

```

```
p. rewrite f, g. apply A. assert (a' * a # a' * (rngunell1 (X := A)
)) as q'. assert ((rngunell1 (X := A)) ~> (a' * (rngunell1 (X := A)
))) as f. apply pathsinv0. apply (rngmultx0 A). rewrite
<- f. assumption. apply ((pr1 (acommrng_amult A)) a').
). assumption. Defined.
```

```
Definition afldtoaintdom (A : afld) : aintdom . Proof. intro
. split with (pr1 A) . split. apply A. intros a b p q. apply
afldinvertibletoazero. apply multinvmultstable. apply A. assumption.
apply A. assumption. Defined.
```

```
Lemma timesazero {A : acommrng} {a b : A} (p : a * b # 0) :
dirprod (a # 0) (b # 0). Proof. intros. split. assert (a * b #
0 * b) as h. rewrite (rngmult0x A). assumption. apply ((pr2 (
acommrng_amult A)) b). assumption. apply ((pr1 (acommrng_amult
A)) a). rewrite (rngmultx0 A). assumption. Defined.
```

```
Lemma aaminuszero {A : acommrng} {a b : A} (p : a # b) : (a - b
) # 0. Proof. intros. rewrite <- (rngrunax1 A a) in p. rewrite <-
(rngrunax1 A b) in p. assert (a + 0 ~> (a + (b - b))) as
f. rewrite <- (rngrinvox1 A b). apply idpath. rewrite f in
p. rewrite <- (rngmultwithminus1 A a) in p. rewrite <- (rngassoc1 A)
in p. rewrite (rngcomm1 A a) in p. rewrite (rngassoc1 A b) in
p. rewrite (rngmultwithminus1 A a) in p. apply ((pr1 (
acommrng_aadd A)) b (a - b) 0). assumption. Defined.
```

```
Lemma aminuszeroa {A : acommrng} {a b : A} (p : (a - b) # 0) :
a # b. Proof. intros. change 0 with (@rngunell1 A) in p. rewrite
<- (rngrinvox1 A b) in p. rewrite <- (rngmultwithminus1 A a) in
p. apply ((pr2 (acommrng_aadd A)) (-1 * b) a b). assumption.
Defined.
```

Close Scope rng_scope.

(** * VI. Lemmas on logic *)

```
Lemma horelim (A B : UU0) (P : hProp) : dirprod (ishinh_UU A -> P)
(ishinh_UU B -> P) -> (hdisj A B -> P). Proof. intros A B P
p. intro q. apply q. intro u. destruct u as [u | v]. apply (pr1 p
). intro Q. auto. apply (pr2 p). intro Q. auto. Defined.
```

```
Lemma stronginduction {E : nat -> UU} (p : E 0%nat) (q : forall n
: nat, natneq n 0%nat -> ((forall m : nat, natlth m n -> E m) -> E
n)) : forall n : nat, E n. Proof. intros. destruct
n. assumption. apply q. apply (negpathssx0 n). induction n. intros
m t. rewrite (natlthitois0 m t). assumption. intros m t. destruct
(natlehchoice _ _ (natlthstoileh _ _ t)) as [left | right].
apply IHn. assumption. apply q. rewrite right. intro f. apply (
negpathssx0 n). assumption. intros k s. rewrite right in s. apply (
IHn k). assumption. Defined.
```

```
Lemma setquotprpathsandR {X : UU0} (R : eqrel X) : forall x y : X,
setquotpr R x ~> setquotpr R y -> R x y. Proof. intros. assert (pr1
(setquotpr R x) y) as i. assert (pr1 (setquotpr R y) y) as
i0. unfold setquotpr. apply R. destruct X0. assumption. apply i.
Defined.
```

(* Some lemmas on decidable properties of natural numbers. *)

```
Definition isdecnatprop (P : nat -> hProp) := forall m : nat, coprod
(P m) (neg (P m)).
```

```
Lemma negisdecnatprop (P : nat -> hProp) (is : isdecnatprop P) :
isdecnatprop (fun n : nat => hnec (P n)). Proof. intros P is
n. destruct (is n) as [l | r]. apply ii2. intro j. assert hfals
e as x. apply j. assumption. apply x. apply i1. assumption. Defined.
```

```
Lemma bndexistsisdecnatprop (P : nat -> hProp) (is : isdecnatprop P) :
isdecnatprop (fun n : nat => hexists (fun m : nat => dirprod (
natleh m n) (P m))) . Proof. intros P is n. induction
n. destruct (is 0%nat) as [l | r]. apply i1. apply
total2tohexists. split with 0%nat. split. apply
isrefinatleh. assumption. apply ii2. intro j. assert hfals
e as x. apply j. intro m. destruct m as [m m']. apply r. rewrite (
natleh0tois0 m (pr1 m')) in m'. apply m'. apply x.
```

```
destruct (is (S n)) as [l | r]. apply i1. apply
total2tohexists. split with (S n). split. apply (isrefinatleh (S n
)). assumption. destruct IHn as [l' | r']. apply i1. apply l'.
intro m. destruct m as [m m']. apply total2tohexists. split with
m. split. apply (istransnatleh _ n _). apply m'. apply
natlthtoleh. apply natlthnsn. apply m'. apply ii2. intro j. apply
r'. apply j. intro m. destruct m as [m m']. apply
total2tohexists. split with m. split. destruct (natlehchoice m (S n
) (pr1 m')). apply natlthstoileh. assumption. assert empty. apply
r. rewrite <- i. apply m'. contradiction. apply m'. Defined.
```

```
Lemma isdecisbndqdec (P : nat -> hProp) (is : isdecnatprop P) (n
: nat) : coprod (forall m : nat, natleh m n -> P m) (hexists (fun
m : nat => dirprod (natleh m n) (neg (P m)))) . Proof. intros
P is n. destruct (bndexistsisdecnatprop _ (negisdecnatprop P is) n
) as [l | r]. apply ii2. assumption. apply i1. intros m j. destruct
(is m) as [l' | r']. assumption. assert hfals
e as x. apply
r. apply total2tohexists. split with
m. split. assumption. assumption. contradiction. Defined.
```

```
Lemma leastelementprinciple (n : nat) (P : nat -> hProp) (is :
isdecnatprop P) : P n -> hexists (fun k : nat => dirprod (P k) (
forall m : nat, natlth m k -> neg (P m))) . Proof. intro
n. induction n. intros P is u. apply total2tohexists. split with
0%nat. split. assumption. intros m i. assert empty. apply (
negnatgth0n m i). contradiction. intros P is u. destruct (is 0%nat
) as [l | r]. apply total2tohexists. split with 0%nat. split.
assumption. intros m i. assert empty. apply (negnatgth0n m i
). contradiction. set (P' := fun m : nat => P (S m)). assert (
forall m : nat, coprod (P' m) (neg (P' m))) as is'. intros
m. unfold P'. apply (is (S m)). set (c := IHn P' is' u). apply
c. intros k. destruct k as [k v]. destruct v as [v0 v1]. apply
total2tohexists. split with (S k). split. assumption. intros
m. destruct m. intros i. assumption. intros i. apply v1. apply i.
Defined.
```

(** END OF FILE *)

7.4 The file fps.v

```

(** *Formal Power Series *)

(** By Alvaro Pelayo, Vladimir Voevodsky and Michael A. Warren *)

(** January 2011 *)

(** Settings *)

Add Rec LoadPath "../Generalities". Add Rec LoadPath "../hlevel1".
Add Rec LoadPath "../hlevel2". Add Rec LoadPath
"../Proof_of_Extensionality". Add Rec LoadPath "../Algebra".

Unset Automatic Introduction. (** This line has to be removed for the
file to compile with Coq8.2 *)

(** Imports *)

Require Export lemmas.

(** ** I. Summation in a commutative ring *)

Open Scope rng_scope.

Definition natsummation0 { R : commrng } ( upper : nat ) ( f : nat -> R ) : R. Proof. intro R. intro upper. induction upper. intros. exact
( f 0%nat ). intros. exact ( ( IHupper f + ( f ( S upper ) ) ) ).
Defined.

Lemma natsummationpaths { R : commrng } { upper upper' : nat } ( u :
upper "> upper' ) ( f : nat -> R ) : natsummation0 upper f ">
natsummation0 upper' f. Proof. intros. destruct u. auto. Defined.

Lemma natsummationpathsupperfixed { R : commrng } { upper : nat } ( f
f' : nat -> R ) ( p : forall x : nat, natleh x upper -> f x "> f' x )
: natsummation0 upper f "> natsummation0 upper f'. Proof. intros R
upper. induction upper. intros f f' p. simpl. apply p. apply
isreflnatleh. intros. simpl. rewrite ( IHupper f f' ). rewrite ( p (
S upper ) ). apply idpath. apply isreflnatleh. intros x p'. apply
p. apply ( istransnatleh _ upper ). assumption. apply
natlthtoleh. apply natlthnsn. Defined.

(* Here we consider summation of functions which are, in a fixed
interval, 0 for all but either the first or last value. *)

Lemma natsummationae0bottom { R : commrng } { f : nat -> R } ( upper :
nat ) ( p : forall x : nat, natlth 0 x -> f x "> 0 ) : natsummation0
upper f "> ( f 0%nat ). Proof. intros R f upper. induction
upper. auto. intro p. simpl. rewrite ( IHupper ). rewrite ( p ( S
upper ) ). rewrite ( rngrunax1 R ). apply idpath. apply (
natlehlthtrans _ upper _ ). apply natleh0n. apply
natlthnsn. assumption. Defined.

Lemma natsummationae0top { R : commrng } { f : nat -> R } ( upper :
nat ) ( p : forall x : nat, natlth x upper -> f x "> 0 ) :
natsummation0 upper f "> ( f upper ). Proof. intros R f
upper. induction upper. auto. intro p. assert ( natsummation0 upper f
"> ( natsummation0 ( R := R ) upper ( fun x : nat => 0 ) ) ) as g.
apply natsummationpathsupperfixed. intros m q. apply p. exact (
natlehlthtrans m upper ( S upper ) q ( natlthnsn upper ) ).
simpl. rewrite g. assert ( natsummation0 ( R := R ) upper ( fun _ :
nat => 0 ) "> 0 ) as g'. set ( g'' := fun x : nat => rngunell ( X := R
) ). assert ( forall x : nat, natlth 0 x -> g'' x "> 0 ) as q0. intro
k. intro pp. auto. exact ( natsummationae0bottom upper q0 ). rewrite
g'. rewrite ( rnglunax1 R ). apply idpath. Defined.

Lemma natsummationshift0 { R : commrng } ( upper : nat ) ( f : nat ->
R ) : natsummation0 ( S upper ) f "> ( natsummation0 upper ( fun x :
nat => f ( S x ) ) + f 0%nat ). Proof. intros R upper. induction
upper. intros f. simpl. apply R. intros. change ( natsummation0 ( S
upper ) f + f ( S ( S upper ) ) "> ( natsummation0 upper ( fun x : nat
=> f ( S x ) ) + f ( S ( S upper ) ) + f 0%nat ) ). rewrite
IHupper. rewrite 2! ( rngassoc1 R ). rewrite ( rngcomm1 R ( f 0%nat )
_ ). apply idpath. Defined.

Lemma natsummationshift { R : commrng } ( upper : nat ) ( f : nat -> R
) { i : nat } ( p : natleh i upper ) : natsummation0 ( S upper ) f ">
( natsummation0 upper ( funcomp ( natcoface i ) f ) + f i ). Proof.
intros R upper. induction upper. intros f i p. destruct i. unfold
funcomp. apply R. assert empty. exact ( negnatlehsn0 i p
). contradiction. intros f i p. destruct i. apply natsummationshift0.
destruct ( natlehchoice ( S i ) ( S upper ) p ) as [ h | k ]. change
( natsummation0 ( S upper ) f + f ( S ( S upper ) ) "> ( natsummation0
( S upper ) ( funcomp ( natcoface ( S i ) ) f ) + f ( S i ) )
). rewrite ( IHupper f ( S i ) ). simpl. unfold funcomp at 3. unfold
natcoface at 3. rewrite 2! ( rngassoc1 R ). rewrite ( rngcomm1 R _ (
f ( S i ) ) ). simpl. rewrite ( natgehimplnatgtbfalse i upper ). apply
idpath. apply p. apply natlthnsntoleh. assumption. simpl. assert (
natsummation0 upper ( funcomp ( natcoface ( S i ) ) f ) ">
natsummation0 upper f ) as h. apply
natsummationpathsupperfixed. intros m q. unfold funcomp. unfold
natcoface. assert ( natlth m ( S i ) ) as q'. apply ( natlehlthtrans _
upper ). assumption. rewrite k. apply natlthnsn. unfold natlth in q'.
rewrite q'. apply idpath. rewrite <- h. unfold funcomp, natcoface at
3. simpl. rewrite ( natgehimplnatgtbfalse i upper ). rewrite 2! (
rngassoc1 R ). rewrite ( rngcomm1 R ( f ( S ( S upper ) ) ) ). rewrite
k. apply idpath. apply p. Defined.

Lemma natsummationplusdistr { R : commrng } ( upper : nat ) ( f g :
nat -> R ) : natsummation0 upper ( fun x : nat => f x + g x ) "> ( (
natsummation0 upper f ) + ( natsummation0 upper g ) ). Proof. intros
R upper. induction upper. auto. intros f g. simpl. rewrite <- (
rngassoc1 R _ ( natsummation0 upper g ) _ ). rewrite ( rngassoc1 R (
natsummation0 upper f ) ). rewrite ( rngcomm1 R _ ( natsummation0
upper g ) ). rewrite <- ( rngassoc1 R ( natsummation0 upper f ) ).
rewrite <- ( IHupper f g ). rewrite ( rngassoc1 R ). apply idpath.
Defined.

Lemma natsummationtimesdistr { R : commrng } ( upper : nat ) ( f : nat
-> R ) ( k : R ) : k * ( natsummation0 upper f ) "> ( natsummation0
upper ( fun x : nat => k * f x ) ). Proof. intros R upper. induction
upper. auto. intros f k. simpl. rewrite <- ( IHupper ). rewrite <- (
rngldistr R ). apply idpath. Defined.

Lemma natsummationtimesdistl { R : commrng } ( upper : nat ) ( f : nat
-> R ) ( k : R ) : ( natsummation0 upper f ) * k "> ( natsummation0
upper ( fun x : nat => f x * k ) ). Proof. intros R upper. induction
upper. auto. intros f k. simpl. rewrite <- IHupper. rewrite (
rngrdistr R ). apply idpath. Defined.

Lemma natsummationsswapminus { R : commrng } { upper n : nat } ( f :
nat -> R ) ( q : natleh n upper ) : natsummation0 ( S ( minus upper n
) ) f "> natsummation0 ( minus ( S upper ) n ) f. Proof. intros R
upper. induction upper. intros n f q. destruct n. auto. assert
empty. exact ( negnatlehsn0 n q ). contradiction. intros n f
q. destruct n. auto. change ( natsummation0 ( S ( minus upper n ) ) f
"> natsummation0 ( minus ( S upper ) n ) f ). apply IHupper. apply q.
Defined.

(** The following lemma asserts that

$$\sum_{k=0}^n \sum_{l=0}^k f(1, k-l) = \sum_{k=0}^n \sum_{l=k}^n f(n-k, l)$$

*)

```

Lemma natsummationswap { R : commrng } (upper : nat) (f : nat → nat → R) : natsummation0 upper (fun i : nat => natsummation0 i (fun j : nat => f j (minus i j))) ~> (natsummation0 upper (fun k : nat => natsummation0 (minus upper k) (fun l : nat => f k l))) .
Proof. intros R upper. induction upper. auto.

```

  intros f. change ( natsummation0 upper ( fun i : nat => natsummation0
    i ( fun j : nat => f j ( minus i j ) ) ) + natsummation0 ( S upper ) (
      fun j : nat => f j ( minus ( S upper ) j ) ) ) ~> ( natsummation0
        upper ( fun k : nat => natsummation0 ( S upper - k ) ( fun l : nat => f
          k l ) ) + natsummation0 ( minus ( S upper ) ( S upper ) ) ( fun l :
            nat => f ( S upper ) l ) ) ) . change ( natsummation0 upper ( fun i :
              nat => natsummation0 i ( fun j : nat => f j ( minus i j ) ) ) + (
                natsummation0 upper ( fun j : nat => f j ( minus ( S upper ) j ) ) +
                  f ( S upper ) ( minus ( S upper ) ( S upper ) ) ) ~> ( natsummation0
                    upper ( fun k : nat => natsummation0 ( S upper - k ) ( fun l : nat => f
                      k l ) ) + natsummation0 ( minus ( S upper ) ( S upper ) ) ( fun l :
                        nat => f ( S upper ) l ) ) ) .

```

```

  assert ( ( natsummation0 upper ( fun k : nat => natsummation0 ( S (
    minus upper k ) ) ) ( fun l : nat => f k l ) ) ) ~> ( natsummation0 upper
      ( fun k : nat => natsummation0 ( minus ( S upper ) k ) ( fun l : nat =>
        f k l ) ) ) ) as A. apply natsummationpathsupperfixed. intros n
    q. apply natsummationsswapminus. exact q. rewrite <- A. change (
      fun k : nat => natsummation0 ( S ( minus upper k ) ) ( fun l : nat => f
        k l ) ) with ( fun k : nat => natsummation0 ( minus upper k ) ) ( fun l
          : nat => f k l ) + f k ( S ( minus upper k ) ) ) . rewrite (
            natsummationplusdistr upper _ ( fun k : nat => f k ( S ( minus upper
              k ) ) ) ) . rewrite IHupper. rewrite minusnn0. rewrite ( rngassoc1
                R ) . assert ( natsummation0 upper ( fun j : nat => f j ( minus ( S
                  upper ) j ) ) ~> natsummation0 upper ( fun k : nat => f k ( S (
                    minus upper k ) ) ) ) as g. apply
                  natsummationpathsupperfixed. intros m q. rewrite pathssminus. apply
                    idpath. apply ( natlehlthtrans _ upper ) . assumption. apply
                      natlthnsn. rewrite g. apply idpath. Defined.

```

(** * II. Reindexing along functions i : nat → nat which are automorphisms of the interval of summation. *)

Definition isnatruncauto (upper : nat) (i : nat → nat) :=
dirprod (forall x : nat, natleh x upper → total2 (fun y : nat =>
dirprod (natleh y upper) (dirprod (i y ~> x) (forall z : nat,
natleh z upper → i z ~> x → y ~> z)))) (forall x : nat, natleh
x upper → natleh (i x) upper) .

Lemma natruncautoisinj { upper : nat } { i : nat → nat } (p :
isnatruncauto upper i) { n m : nat } (n' : natleh n upper) (m' :
natleh m upper) : i n ~> i m → n ~> m. Proof. intros upper i p n m
n' m' h. assert (natleh (i m) upper) as q. apply
p. assumption. set (x := pr1 p (i m) q) . set (v := pr1 x) . set (w := pr1 (pr2 x)) . set (y := pr1 (pr2 (pr2 x))) . change (pr1 x) with v in w, y. assert (v ~> n) as a. apply (pr2 x) . assumption. assumption. rewrite <- a. apply (pr2 x) . assumption. apply idpath. Defined.

Definition natruncautopreimage { upper : nat } { i : nat → nat } (p :
isnatruncauto upper i) { n : nat } (n' : natleh n upper) : nat
:= pr1 (pr1 p n n') .

Definition natruncautopreimagepath { upper : nat } { i : nat → nat } (p :
isnatruncauto upper i) { n : nat } (n' : natleh n upper) : i
(natruncautopreimage p n') ~> n := (pr1 (pr2 (pr2 (pr1 p n n')))) .

Definition natruncautopreimageineq { upper : nat } { i : nat → nat } (p :
isnatruncauto upper i) { n : nat } (n' : natleh n upper) :

natleh (natruncautopreimage p n') upper := ((pr1 (pr2 (pr1 p n n')))) .

Definition natruncautopreimagecanon { upper : nat } { i : nat → nat } (p :
isnatruncauto upper i) { n : nat } (n' : natleh n upper) (m : nat) (m' : natleh m upper) (q : i m ~> n) :
natruncautopreimage p n' ~> m := (pr2 (pr2 (pr2 (pr1 p n n'))))
m m' q.

Definition natruncautoinv { upper : nat } { i : nat → nat } (p :
isnatruncauto upper i) : nat → nat. Proof. intros upper i p
n. destruct (natgthorleh n upper) as [l | r] . exact n. exact (natruncautopreimage p r) . Defined.

Lemma natruncautoinvvisnatruncauto { upper : nat } { i : nat → nat } (p :
isnatruncauto upper i) : isnatruncauto upper (natruncautoinv p) . Proof. intros. split. intros n n' . split with (i n) . split. apply p. assumption. split. unfold
natruncautoinv. destruct (natgthorleh (i n) upper) as [l | r] . assert empty. apply (isirreflnatlth (i n)) . apply (natlehlthtrans _ upper) . apply
p. assumption. assumption. contradiction. apply (natruncautoisinj p) . apply (natruncautopreimageineq) . assumption. apply (natruncautopreimagepath p r) . intros m x v. unfold natruncautoinv
in v. destruct (natgthorleh m upper) as [l | r] . assert empty. apply (isirreflnatlth upper) . apply (natlthlethtrans _ m) . assumption. assumption. contradiction. rewrite <- v. apply (natruncautopreimagepath p r) . intros x X. unfold
natruncautoinv. destruct (natgthorleh x upper) as [l | r] . assumption. apply (natruncautopreimageineq p r) . Defined.

Definition truncnatruncauto { upper : nat } { i : nat → nat } (p :
isnatruncauto (S upper) i) : nat → nat. Proof. intros upper i p
n. destruct (natlthorgeh (i n) (S upper)) as [l | r] . exact (i n) . destruct (natgehchoice _ _ r) as [a | b] . exact (i n) . destruct (isdeceqnat n (S upper)) as [h | k] . exact (i n) . exact (i (S upper)) . Defined.

Lemma truncnatruncautobound { upper : nat } (i : nat → nat) (p :
isnatruncauto (S upper) i) (n : nat) (q : natleh n upper) :
natleh (truncnatruncauto p n) upper. Proof. intros. unfold
truncnatruncauto. destruct (natlthorgeh (i n) (S upper)) as [l | r] . apply natlthnsntoleh. assumption. destruct (natgehchoice (i n) (S upper)) as [l' | r'] . assert empty. apply (isirreflnatlth (i n)) . apply (natlehlthtrans _ (S upper)) . apply p. apply
natlthtoleh. apply (natlehlthtrans _ upper) . assumption. apply
natlthnsn. assumption. contradiction. destruct (isdeceqnat n (S upper)) as [l' | r'] . assert empty. apply (isirreflnatlth upper) . apply (natlthlethtrans _ (S upper)) . apply natlthnsn. rewrite <- l' . assumption. contradiction. assert (natleh (i (S upper)) (S upper)) as aux. apply p. apply isreflnatlth. destruct (natlechoice _ _ aux) as [l' | r'] . apply natlthnsntoleh. assumption. assert empty. apply r' . apply (natruncautoisinj p) . apply
natlthtoleh. apply (natlehlthtrans _ upper) . assumption. apply
natlthnsn. apply isreflnatlth. rewrite r' . rewrite r' . apply
idpath. contradiction. Defined.

Lemma truncnatruncautoisinj { upper : nat } { i : nat → nat } (p :
isnatruncauto (S upper) i) { n m : nat } (n' : natleh n upper) (m' :
natleh m upper) : truncnatruncauto p n ~> truncnatruncauto p m → n ~> m. Proof. intros upper i p n m q r s. apply (natruncautoisinj p) . apply natlthtoleh. apply (natlehlthtrans _ upper) . assumption. apply natlthnsn. apply natlthtoleh. apply (natlehlthtrans _ upper) . assumption. apply natlthnsn. unfold
truncnatruncauto in s. destruct (natlthorgeh (i n) (S upper)) as [a0 | a1] . destruct (natlthorgeh (i m) (S upper)) as [b0 |

```

b1 ]. assumption. assert empty. assert ( i m ~> S upper ) as f0.
destruct ( natgehchoice ( i m ) ( S upper ) b1 ) as [ l | l' ]. assert
empty. apply ( isirreflnatlth ( S upper ) ). apply ( natlehlthtrans _
( i n ) ). rewrite
s. assumption. assumption. contradiction. assumption. destruct (
natgehchoice ( i m ) ( S upper ) b1 ) as [ a0 | a10 ]. apply (
isirreflnatgth ( S upper ) ). rewrite f0 in a0. assumption. destruct
( isdeceqnat m ( S upper ) ) as [ a00 | a100 ]. rewrite s in
a0. rewrite f0 in a0. apply ( isirreflnatlth ( S upper )
). assumption. assert ( i m ~> n ) as f1. apply ( natruncautoisinj p
). rewrite f0. apply isreflnatleh. apply natlthtoleh. apply (
natlehlthtrans _ upper ). assumption. apply natlthnsn. rewrite f0.
rewrite s. apply idpath. apply ( isirreflnatlth upper ). apply (
natlthlehtans _ n ). rewrite <- f1, f0. apply
natlthnsn. assumption. contradiction. destruct ( natgehchoice ( i n )
( S upper ) a1 ) as [ a00 | a01 ]. assert empty. apply (
isirreflnatlth ( S upper ) ). apply ( natlthlehtans _ ( i n )
). assumption. apply ( p ). apply natlthtoleh. apply ( natlehlthtrans
_ upper ). assumption. apply natlthnsn. contradiction. destruct (
natlthorgeh ( i m ) ( S upper ) ) as [ b0 | b1 ]. destruct (
isdeceqnat n ( S upper ) ) as [ a000 | a001 ]. assumption. assert ( S
upper ~> m ) as f0. apply ( natruncautoisinj p ). apply
isreflnatleh. apply natlthtoleh. apply ( natlehlthtrans _ upper
). assumption. apply natlthnsn. assumption. assert empty. apply
a001. rewrite f0. assert empty. apply ( isirreflnatlth ( S upper )
). apply ( natlehlthtrans _ upper ). rewrite f0. assumption. apply
natlthnsn. contradiction. contradiction. destruct ( natgehchoice ( i
m ) ( S upper ) b1 ) as [ b00 | b01 ]. assert empty. apply (
isirreflnatlth ( i m ) ). apply ( natlehlthtrans _ ( S upper )
). apply p. apply ( natlthtoleh ). apply ( natlehlthtrans _ upper
). assumption. apply natlthnsn. assumption. contradiction. rewrite
b01. rewrite a01. apply idpath. Defined.

```

```

Lemma truncnatruncautoisauto { upper : nat } { i : nat -> nat } ( p :
isnatruncauto ( S upper ) i ) : isnatruncauto upper (
truncnatruncauto p ). Proof. intros. split. intros n q. assert (
natleh n ( S upper ) ) as q'. apply natlthtoleh. apply (
natlehlthtrans _ upper ). assumption. apply natlthnsn. destruct (
isdeceqnat ( natruncautoimage p q' ) ( S upper ) ) as [ i0 | i1
]. split with ( natruncautoimage p ( isreflnatleh ( S upper ) )
). split. assert ( natleh ( natruncautoimage p ( isreflnatleh ( S
upper ) ) ) ( S upper ) ) as aux. apply
natruncautoimageineq. destruct ( natlehchoice _ _ aux ) as [ l | r
]. apply natlthnsntoleh. assumption. assert ( n ~> S upper ) as
f0. rewrite <- ( natruncautoimagepath p q' ). rewrite i0. rewrite
<- r. rewrite ( natruncautoimagepath p ( isreflnatleh ( S upper ) )
). rewrite r. apply idpath. assert empty. apply ( isirreflnatlth ( S
upper ) ). apply ( natlehlthtrans _ upper ). rewrite <-
f0. assumption. apply natlthnsn. contradiction.

```

```

split. apply ( natruncautoisinj p ). apply natlthtoleh. apply (
natlehlthtrans _ upper ). apply truncnatruncautobound. destruct (
natlehchoice _ _ ( natruncautoimageineq p ( isreflnatleh ( S
upper ) ) ) ) as [ l | r ]. apply natlthnsntoleh. assumption. assert
empty. assert ( S upper ~> n ) as f0. rewrite <- (
natruncautoimagepath p ( isreflnatleh ( S upper ) ) ). rewrite
r. rewrite <- i0. rewrite ( natruncautoimagepath p q' ). apply
idpath. apply ( isirreflnatlth ( S upper ) ). apply ( natlehlthtrans
_ upper ). rewrite f0. assumption. apply
natlthnsn. contradiction. apply natlthnsn. assumption. unfold
truncnatruncauto. destruct ( isdeceqnat ( natruncautoimage p (
isreflnatleh ( S upper ) ) ) ) as [ l | r ]. assert empty. assert (
S upper ~> n ) as f0. rewrite <- ( natruncautoimagepath p (
isreflnatleh ( S upper ) ) ). rewrite l. rewrite <- i0. rewrite (
natruncautoimagepath p q' ). apply idpath. apply (
isirreflnatlth ( S upper ) ). apply ( natlehlthtrans _ upper

```

```

). rewrite f0. assumption. apply natlthnsn. contradiction. destruct
( natlthorgeh ( i ( natruncautoimage p ( isreflnatleh ( S upper
) ) ) ) ( S upper ) ) as [ l' | r' ]. assert empty. apply (
isirreflnatlth ( S upper ) ) . rewrite ( natruncautoimagepath p
) in l'. assumption. contradiction. destruct ( natgehchoice _ _ r' )
as [ l'' | r'' ]. assert empty. apply ( isirreflnatlth ( S upper ) )
. rewrite ( natruncautoimagepath p ) in
l''. assumption. contradiction. rewrite <- i0. rewrite (
natruncautoimagepath p q' ). apply idpath. intros x X y. apply
( natruncautoisinj p ). apply natruncautoimageineq. apply
natlthtoleh. apply ( natlehlthtrans _ upper ). assumption. apply
natlthnsn. unfold truncnatruncauto in y. destruct ( natlthorgeh (
i x ) ( S upper ) ) as [ l | r ]. assert ( S upper ~> x ) as
f0. apply ( natruncautoisinj p ). apply isreflnatleh. apply
natlthtoleh. apply ( natlehlthtrans _ upper ). assumption. apply
natlthnsn. rewrite <- i0. rewrite y. rewrite (
natruncautoimagepath p q' ). apply idpath. assert empty. apply (
isirreflnatlth ( S upper ) ). apply ( natlehlthtrans _ upper
). rewrite f0. assumption. apply natlthnsn. contradiction. destruct
( isdeceqnat x ( S upper ) ) as [ l' | r' ]. assert empty. apply (
isirreflnatlth ( S upper ) ). apply ( natlehlthtrans _ upper
). rewrite <- l'. assumption. apply
natlthnsn. contradiction. destruct ( natgehchoice _ _ r ) as [ l'' |
r'' ]. assert empty. apply ( isirreflnatlth n ). apply (
natlehlthtrans _ ( S upper ) ). assumption. rewrite <-
y. assumption. contradiction. rewrite ( natruncautoimagepath p _
). rewrite r''. apply idpath. split with ( natruncautoimage p
q' ). split. destruct ( natlehchoice _ _ ( natruncautoimageineq
p q' ) ) as [ l | r ]. apply natlthnsntoleh. assumption. assert
empty. apply i1. assumption. contradiction. split. unfold
truncnatruncauto. destruct ( natlthorgeh ( i (
natruncautoimage p q' ) ) ( S upper ) ) as [ l | r ]. apply
natruncautoimagepath. destruct ( natgehchoice _ _ r ) as [ l' |
r' ]. apply natruncautoimagepath. assert empty. apply (
isirreflnatlth ( S upper ) ). apply ( natlehlthtrans _ upper
). rewrite <- r'. rewrite ( natruncautoimagepath p q'
). assumption. apply natlthnsn. contradiction.

```

```

intros x X y. apply ( natruncautoisinj p ). apply ( pr1 p ). apply
natlthtoleh. apply ( natlehlthtrans _ upper ). assumption. apply
natlthnsn. rewrite ( natruncautoimagepath p q' ). unfold
truncnatruncauto in y. destruct ( natlthorgeh ( i x ) ( S upper ) )
as [ l | r ]. rewrite y. apply idpath. destruct ( isdeceqnat x ( S
upper ) ) as [ l' | r' ]. assert empty. apply ( isirreflnatlth ( S
upper ) ). apply ( natlehlthtrans _ upper ). rewrite <-
l'. assumption. apply natlthnsn. contradiction. destruct (
natgehchoice _ _ r ). rewrite y. apply idpath. assert empty. apply
i1. apply ( natruncautoisinj p ). apply ( natruncautoimageineq
p ). apply isreflnatleh. rewrite ( natruncautoimagepath p q'
). rewrite y. apply idpath. contradiction. apply
truncnatruncautobound. Defined.

```

```

Definition truncnatruncautoinv { upper : nat } { i : nat -> nat } ( p
: isnatruncauto ( S upper ) i ) : nat -> nat := natruncautoinv (
truncnatruncautoisauto p ).

```

```

Lemma precompwithnatcofaceisauto { upper : nat } ( i : nat -> nat ) (
p : isnatruncauto ( S upper ) i ) ( bound : natlth 0 (
natruncautoimage p ( isreflnatleh ( S upper ) ) ) ) :
isnatruncauto upper ( funcomp ( natcoface ( natruncautoimage p (
isreflnatleh ( S upper ) ) ) ) i ). Proof. intros. set ( v :=
natruncautoimage p ( isreflnatleh ( S upper ) ) ). change (
natruncautoimage p ( isreflnatleh ( S upper ) ) ) with v in
bound. unfold isnatruncauto. split. intros m q. unfold
funcomp. assert ( natleh m ( S upper ) ) as aaa. apply
natlthtoleh. apply natlehlthtrans with ( m := upper

```

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). assumption. exact ( natlthnsn upper ). set ( m' :=
natruncautopreimage p aaa ). destruct ( natlthorgeh m' v ) as [ l |
r ]. ( * CASE m' < v * ) split with m'. split. apply natlthnsntoleh.
apply ( natlthlehtans _ v ). assumption. apply (
natruncautopreimageineq p _ ). split. unfold natcoface. rewrite
l. apply ( natruncautopreimagepath p aaa ). intros n j w. assert (
natcoface v n "> m' ) as f0. apply pathsinv0. apply (
natruncautopreimagecanon p aaa ). apply
natcofaceleh. assumption. assumption. rewrite <- f0. destruct (
natgthorleh v n ) as [ l' | r' ]. unfold natcoface. rewrite l'. apply
idpath. assert empty. apply ( isirreflnatlth v ). apply (
natlehlthtrans _ n ). assumption. apply ( istransnatlth _ ( S n )
). apply natlthnsn. unfold natcoface in f0. rewrite (
natgehimplnatgtbfalse v n r' ) in f0. rewrite
f0. assumption. contradiction. ( * CASE v <= m' * ) set ( j :=
natruncautopreimagepath p aaa ). change ( natruncautopreimage p aaa
) with m' in j. set ( m'' := minus m' 1 ). assert ( natleh m'' upper
) as a0. destruct ( natlthorgeh 0 m' ) as [ h | h' ]. rewrite <- (
minussn1 upper ). apply minuslelh. assumption. apply ( natlehlthtrans
_ upper ). apply natleh0n. apply natlthnsn. apply (
natruncautopreimageineq ). destruct ( natgehchoice 0 m' h' ) as [ k |
k' ]. assert empty. apply ( negnatgth0n m' k ). contradiction. unfold
m''. rewrite <- k'. apply natleh0n. destruct ( natgehchoice m' v r )
as [ l' | r' ]. assert ( natleh v m'' ) as a2. apply
natlthnsntoleh. unfold m''. rewrite pathssminus. rewrite
minussn1. assumption. destruct ( natlehchoice 0 m' ( natleh0n m' ) )
as [ k | k' ]. assumption. assert empty. apply ( negnatgth0n v
). rewrite k'. assumption. contradiction. assert ( i ( natcoface v
m'' ) "> m ) as a1. unfold natcoface. rewrite ( natgehimplnatgtbfalse
v m'' a2 ). unfold m''. rewrite pathssminus. rewrite
minussn1. assumption. destruct ( natlehchoice 0 m' ( natleh0n m' ) )
as [ k | k' ]. assumption. assert empty. apply ( negnatgth0n v
). rewrite k'. assumption. contradiction. split with
m''. split. assumption. split. assumption. intros n s t. assert (
natcoface v n "> natcoface v m'' ) as g. assert ( natcoface v n "> m'
) as g0. apply pathsinv0. apply ( natruncautopreimagecanon p aaa
). apply natcofaceleh. assumption. assumption. assert ( natcoface v
m'' "> m' ) as g1. unfold m'. unfold natruncautopreimage. apply
pathsinv0. apply ( natruncautopreimagecanon p aaa ). apply
natcofaceleh. assumption. assumption. rewrite g0, g1. apply idpath.
change ( idfun _ m'' "> idfun _ n ). rewrite <- (
natcofaceretractisretract v ). unfold funcomp. rewrite g. apply
idpath. assert empty. apply ( isirreflnatlth ( S upper ) ). apply (
natlehlthtrans _ upper ). assert ( S upper "> m ) as g. rewrite <- (
natruncautopreimagepath p ( isreflnatlth ( S upper ) ) ). change ( i
v "> m ). rewrite <- j. rewrite r'. apply idpath. rewrite
g. assumption. apply natlthnsn. contradiction.

```

```

intros x X. unfold funcomp. assert ( natleh ( i ( natcoface v x
) ) ( S upper ) ) as a0. apply p. apply
natcofaceleh. assumption. destruct ( natlehchoice _ _ a0 ) as [
l | r ]. apply natlthnsntoleh. assumption. assert ( v ">
natcoface v x ) as g. unfold v. apply (
natruncautopreimagecanon p ( isreflnatlth ( S upper ) )
). unfold natcoface. destruct ( natgthorleh v x ) as [ a | b
]. unfold v in a. rewrite a. apply natlthtolen. apply (
natlehlthtrans _ upper ). assumption. apply natlthnsn. unfold v
in b. rewrite ( natgehimplnatgtbfalse _ x b ). assumption.
assumption. assert empty. destruct ( natgthorleh v x ) as [ a |
b ]. unfold natcoface in g. rewrite a in g. apply (
isirreflnatlth x ). rewrite g in a. assumption. unfold natcoface
in g. rewrite ( natgehimplnatgtbfalse v x b ) in g. apply (
isirreflnatlth x ). apply ( natlthlehtans _ ( S x ) ). apply
natlthnsn. rewrite <- g. assumption. contradiction. Defined.

```

Lemma natruncautocompstable { R : commrng } { upper : nat } { i j :

```

nat -> nat ) ( p : isnatruncauto upper i ) ( p' : isnatruncauto
upper j ) : isnatruncauto upper ( funcomp j i ). Proof.
intros. split. intros n n'. split with ( natruncautopreimage p' (
natruncautopreimageineq p n' ) ). split. apply (
natruncautopreimageineq p' ). split. unfold funcomp. rewrite (
natruncautopreimagepath p' _ ). rewrite ( natruncautopreimagepath p
_ ). apply idpath. intros x X y. unfold funcomp in y. apply (
natruncautoisinj p' ). apply
natruncautopreimageineq. assumption. apply ( natruncautoisinj p
). apply p'. apply natruncautopreimageineq. apply
p'. assumption. rewrite ( natruncautopreimagepath p' ). rewrite (
natruncautopreimagepath p ). rewrite y. apply idpath. intros x
X. unfold funcomp. apply p. apply p'. assumption. Defined.

```

Definition natruncreverse (upper : nat) : nat -> nat. Proof.
intros upper n. destruct (natgthorleh n upper) as [h | k]. exact
n. exact (minus upper n). Defined.

Definition natrunccbottomtopswap (upper : nat) : nat -> nat. Proof.
intros upper n. destruct (isdeceqnat 0%nat n) as [h | k]. exact (
upper). destruct (isdeceqnat upper n) as [l | r]. exact (0%nat
). exact n. Defined.

Lemma natruncreverseisnatruncauto (upper : nat) : isnatruncauto
upper (natruncreverse upper). Proof. intros. unfold
isnatruncauto. split. intros m q. set (m' := minus upper m
). assert (natleh m' upper) as a0. apply minuslelh. assert (
natruncreverse upper m' "> m) as a1. unfold
natruncreverse. destruct (natgthorleh m' upper). assert
empty. apply isirreflnatlth with (n := m'). apply natlehlthtrans
with (m := upper). assumption. assumption. contradiction. unfold
m'. rewrite doubleminuslehtpaths. apply idpath. assumption. split with
m'. split. assumption. split. assumption. intros n qq u. unfold
m'. rewrite <- u. unfold natruncreverse. destruct (natgthorleh n
upper) as [l | r]. assert empty. apply (isirreflnatlth n). apply
(natlehlthtrans _ upper
). assumption. assumption. contradiction. rewrite
doubleminuslehtpaths. apply idpath. assumption. intros x X. unfold
natruncreverse. destruct (natgthorleh x upper) as [l | r
]. assumption. apply minuslelh. Defined.

Lemma natrunccbottomtopswapselfinv (upper n : nat) :
natrunccbottomtopswap upper (natrunccbottomtopswap upper n) "> n.
Proof. intros. unfold natrunccbottomtopswap. destruct (isdeceqnat
upper n). destruct (isdeceqnat 0%nat n). destruct (isdeceqnat
0%nat upper). rewrite <- i0. rewrite <- i1. apply idpath. assert
empty. apply e. rewrite i0. rewrite i. apply idpath. contradiction.
destruct (isdeceqnat 0%nat 0%nat). assumption. assert empty. apply
e0. auto. contradiction. destruct (isdeceqnat 0%nat n). destruct (
isdeceqnat 0%nat upper). rewrite <- i. rewrite i0. apply idpath.
destruct (isdeceqnat upper upper). assumption. assert empty. apply
e1. auto. contradiction. destruct (isdeceqnat 0%nat n). assert
empty. apply e0. assumption. contradiction. destruct (isdeceqnat
upper n). assert empty. apply e. assumption. contradiction. auto.
Defined.

Lemma natrunccbottomtopswapbound (upper n : nat) (p : natleh n
upper) : natleh (natrunccbottomtopswap upper n) upper. Proof.
intros. unfold natrunccbottomtopswap. destruct (isdeceqnat 0%nat n
). auto. destruct (isdeceqnat upper n). apply isreflnatlth. apply
isreflnatlth. destruct (isdeceqnat upper n). apply
natleh0n. assumption. Defined.

Lemma natrunccbottomtopswapisnatruncauto (upper : nat) :
isnatruncauto upper (natrunccbottomtopswap upper). Proof.
intros. unfold isnatruncauto. split. intros m p. set (m' :=

```

nattruncbottomtopswap upper m ). assert ( natleh m' upper ) as
a0. apply nattruncbottomtopswapbound. assumption. assert
(nattruncbottomtopswap upper m' ^> m) as a1. apply
nattruncbottomtopswapselfinv. split with
m'. split. assumption. split. assumption. intros k q u. unfold
m'. rewrite <- u. rewrite nattruncbottomtopswapselfinv. apply
idpath. intros n p. apply nattruncbottomtopswapbound. assumption.
Defined.

```

```

Lemma isnatruncauto0S { upper : nat } { i : nat -> nat } ( p :
isnatruncauto (S upper) i ) ( j : i 0%nat ^> S upper ) :
isnatruncauto upper ( funcomp S i ). Proof. intros. unfold
isnatruncauto. split. intros m q. set ( v := natruncautopreimage p
(natlthtoleh m (S upper) (natlehlthtrans m upper (S upper) q
(natlthnsn upper))))). destruct ( isdeceqnat 0%nat v ) as [ i0 | i1
]. assert empty. apply ( isirreflnatlth ( i 0%nat ) ). apply (
natlehlthtrans _ upper ). rewrite i0. unfold v. rewrite (
natruncautopreimagepath ). assumption. rewrite j. apply
natlthnsn. contradiction. assert ( natlh 0 v ) as aux. destruct (
natlehchoice _ _ ( natleh0n v ) ). assumption. assert empty. apply
i1. assumption. contradiction. split with ( minus v 1
). split. rewrite <- ( minusn1 upper ). apply ( minus1leh aux (
natlehlthtrans _ _ _ ( natleh0n upper ) ( natlthnsn upper ) ) )
natruncautopreimageineq p ( natlthtoleh m ( S upper ) (
natlehlthtrans m upper ( S upper ) q ( natlthnsn upper ) ) )
). split. unfold funcomp. rewrite pathssminus. rewrite
minusn1. apply natruncautopreimagepath. assumption. intros n uu
k. unfold funcomp in k. rewrite <- ( minusn1 n ). assert ( v ^> S n
) as f. apply ( natruncautopreimagecanon p _
). assumption. assumption. rewrite f. apply idpath. intros x
X. unfold funcomp. assert ( natleh ( i ( S x ) ) ( S upper ) ) as
aux. apply p. assumption. destruct ( natlehchoice _ _ aux ) as [ h | k
]. apply natlthntoleh. assumption. assert empty. assert ( 0%nat ^> S
x ) as ii. apply ( natruncautoisinj p ). apply natleh0n. assumption.
rewrite j. rewrite k. apply idpath. apply ( isirreflnatlth ( S x ) ).
apply ( natlehlthtrans _ x ). rewrite <- ii. apply natleh0n. apply
natlthnsn. contradiction. Defined.

```

(* The following lemma says that we may reindex sums along automorphisms of the interval over which the finite summation is being taken. *)

```

Lemma natsummationreindexing { R : commrng } { upper : nat } ( i : nat
-> nat ) ( p : isnatruncauto upper i ) ( f : nat -> R ) :
natsummation0 upper f ^> natsummation0 upper ( funcomp i f ). Proof.
intros R upper. induction upper. intros. simpl. unfold funcomp.
assert ( 0%nat ^> i 0%nat ) as f0. destruct ( natlehchoice ( i 0%nat )
0%nat ( pr2 p 0%nat ( isreflnatlth 0%nat ) ) ) as [ h | k ]. assert
empty. exact ( negnatlthn0 ( i 0%nat ) h ). contradiction. rewrite
k. apply idpath. rewrite <- f0. apply idpath. intros. simpl (
natsummation0 ( S upper ) f ). set ( j := natruncautopreimagepath p
( isreflnatlth ( S upper ) ) ). set ( v := natruncautopreimage p (
isreflnatlth ( S upper ) ) ). change ( natruncautopreimage p (
isreflnatlth ( S upper ) ) ) with v in j. destruct ( natlehchoice
0%nat v ( natleh0n v ) ). set ( aaa := natruncautopreimageineq p (
isreflnatlth ( S upper ) ) ). change ( natruncautopreimage p (
isreflnatlth ( S upper ) ) ) with v in aaa. destruct ( natlehchoice v
( S upper ) aaa ) as [ l | r ]. rewrite ( IHupper ( funcomp (
natcoface v ) i ) ).

```

```

change ( funcomp ( funcomp ( natcoface v ) i ) f ) with ( funcomp (
natcoface v ) ( funcomp i f ) ). assert ( f ( S upper ) ^> (
funcomp i f ) v ) as f0. unfold funcomp. rewrite j. apply
idpath. rewrite f0.

```

```

assert ( natleh v upper ) as aux. apply natlthntoleh. assumption.

```

```

rewrite ( natsummationshift upper ( funcomp i f ) aux ). apply
idpath. apply precompwithnatcofaceisauto. assumption.

```

```

rewrite ( IHupper ( funcomp ( natcoface v ) i ) ). assert (
natsummation0 upper ( funcomp ( funcomp ( natcoface v ) i ) f ) ^>
natsummation0 upper ( funcomp i f ) ) as f0. apply
natsummationpathsupperfixed. intros x X. unfold funcomp. unfold
natcoface. assert ( natlh x v ) as a0. apply ( natlehlthtrans _
upper ). assumption. rewrite r. apply natlthnsn. rewrite a0. apply
idpath. rewrite f0. assert ( f ( S upper ) ^> ( funcomp i f ) ( S
upper ) ) as f1. unfold funcomp. rewrite <- r. rewrite j. rewrite
<- r. apply idpath. rewrite f1. apply idpath. apply
precompwithnatcofaceisauto. assumption. rewrite
natsummationshift0. unfold funcomp at 2. rewrite i0. rewrite j.
assert ( i 0%nat ^> S upper ) as j'. rewrite i0. rewrite j. apply
idpath. rewrite ( IHupper ( funcomp S i ) ( isnatruncauto0S p j' )
). apply idpath. Defined.

```

(** * III. Formal Power Series *)

Definition seqson (A : UU) := nat -> A.

Lemma seqsonisaset (A : hSet) : isaset (seqson A). Proof.
intros. unfold seqson. change (isofhlevel 2 (nat -> A)). apply
impredfun. apply A. Defined.

Definition isasetfps (R : commrng) : isaset (seqson R) :=
seqsonisaset R.

Definition fps (R : commrng) : hSet := hSetpair _ (isasetfps R).

Definition fpsplus (R : commrng) : binop (fps R) := fun v w n => (
(v n) + (w n)).

Definition fpstimes (R : commrng) : binop (fps R) := fun s t n =>
natsummation0 n (fun x : nat => (s x) * (t (minus n x))).

(* SOME TESTS OF THE SUMMATION AND FPSTIMES DEFINITIONS:

Definition test0 : seqson hz. Proof. intro n. induction n. exact
0. exact (nattohz (S n)). Defined.

Eval lazy in (hzabsval (natsummation0 1 test0)).

Definition test1 : seqson hz. Proof. intro n. induction n. exact (1
+ 1). exact ((1 + 1) * IHn). Defined.

Eval lazy in (hzabsval (fpstimes hz test0 test1 0%nat)).

Eval lazy in (hzabsval (fpstimes hz test0 test1 1%nat)).

Eval lazy in (hzabsval (fpstimes hz test0 test1 2%nat)).

Eval lazy in (hzabsval (fpstimes hz test0 test1 3%nat)).

Eval lazy in (hzabsval (fpstimes hz test0 test1 4%nat)). *)

Definition fpszero (R : commrng) : fps R := (fun n : nat => 0).

Definition fpsone (R : commrng) : fps R. Proof. intros. intro
n. destruct n. exact 1. exact 0. Defined.

Definition fpsminus (R : commrng) : fps R -> fps R := (fun s n => -
(s n)).

Lemma ismonoidopfpsplus (R : commrng) : ismonoidop (fpsplus R).

Proof. intros. unfold ismonoidop. split. unfold isassoc. intros s t u. unfold fpsplus. (* This is a hack which should work immediately without such a workaround! *) change ((fun n : nat => s n + t n + u n) ~> (fun n : nat => s n + (t n + u n))). apply funextfun. intro n. apply R.

unfold isunital. assert (isunit (fpsplus R) (fpszero R)) as a. unfold isunit. split. unfold islunit. intro s. unfold fpsplus. unfold fpszero. change ((fun n : nat => 0 + s n) ~> s). apply funextfun. intro n. apply rnglunax1.

unfold isrunit. intro s. unfold fpsplus. unfold fpszero. change ((fun n : nat => s n + 0) ~> s). apply funextfun. intro n. apply rngrunax1. exact (tpair (fpszero R) a). Defined.

Lemma isgropfpsplus (R : commrng) : isgrop (fpsplus R). Proof. intros. unfold isgrop. assert (invstruct (fpsplus R) (ismonoidopfpsplus R)) as a. unfold invstruct. assert (isinv (fpsplus R) (unel_is (ismonoidopfpsplus R)) (fpsminus R)) as b. unfold isinv. split. unfold islinv. intro s. unfold fpsplus. unfold fpsminus. unfold unel_is. simpl. unfold fpszero. apply funextfun. intro n. exact (rnglinvax1 R (s n)).

unfold isrinv. intro s. unfold fpsplus. unfold fpsminus. unfold unel_is. simpl. unfold fpszero. apply funextfun. intro n. exact (rngrinvax1 R (s n)). exact (tpair (fpsminus R) b). exact (tpair (ismonoidopfpsplus R) a). Defined.

Lemma iscommfpsplus (R : commrng) : iscomm (fpsplus R). Proof. intros. unfold iscomm. intros s t. unfold fpsplus. change ((fun n : nat => s n + t n) ~> (fun n : nat => t n + s n)). apply funextfun. intro n. apply R. Defined.

Lemma isassocfpstimes (R : commrng) : isassoc (fpstimes R). Proof. intros. unfold isassoc. intros s t u. unfold fpstimes.

assert ((fun n : nat => natsummation0 n (fun x : nat => natsummation0 (minus n x) (fun x0 : nat => s x * (t x0 * u (minus (minus n x) x0)))) ~> (fun n : nat => natsummation0 n (fun x : nat => s x * natsummation0 (minus n x) (fun x0 : nat => t x0 * u (minus (minus n x) x0))))) as A. apply funextfun. intro n. apply natsummationpathsupperfixed. intros. rewrite natsummationtimesdistr. apply idpath. rewrite <- A. assert ((fun n : nat => natsummation0 n (fun x : nat => natsummation0 (minus n x) (fun x0 : nat => s x * t x0 * u (minus (minus n x) x0)))) ~> (fun n : nat => natsummation0 n (fun x : nat => natsummation0 (minus n x) (fun x0 : nat => s x * (t x0 * u (minus (minus n x) x0))))) as B. apply funextfun. intro n. apply maponpaths. apply funextfun. intro m. apply maponpaths. apply funextfun. intro x. apply R. assert ((fun n : nat => natsummation0 n (fun x : nat => natsummation0 x (fun x0 : nat => s x0 * t (minus x x0)) * u (minus n x)))) ~> (fun n : nat => natsummation0 n (fun x : nat => natsummation0 (minus n x) (fun x0 : nat => s x * t x0 * u (minus (minus n x) x0)))) as C. apply funextfun. intro n. set (f := fun x : nat => (fun x0 : nat => s x * t x0 * u (minus (minus n x) x0))). assert (natsummation0 n (fun x : nat => natsummation0 x (fun x0 : nat => f x0 (minus x x0))) ~> (natsummation0 n (fun x : nat => natsummation0 (minus n x) (fun x0 : nat => f x x0)))) as D. apply natsummationswap. unfold f in D. assert (natsummation0 n (fun x : nat => natsummation0 x (fun x0 : nat => s x0 * t (minus x x0) * u (minus n x))) ~> natsummation0 n (fun x : nat => natsummation0 x (fun x0 : nat => s x0 * t (minus x x0)) * u (minus n x))) as E. apply

natsummationpathsupperfixed. intros k p. apply natsummationpathsupperfixed. intros l q. rewrite (natdoubleminus p q). apply idpath. rewrite E, D. apply idpath. rewrite <- B. rewrite <- C. assert ((fun n : nat => natsummation0 n (fun x : nat => natsummation0 x (fun x0 : nat => s x0 * t (minus x x0)) * u (minus n x))) ~> (fun n : nat => natsummation0 n (fun x : nat => natsummation0 x (fun x0 : nat => s x0 * t (minus x x0)) * u (minus n x)))) as D. apply funextfun. intro n. apply maponpaths. apply funextfun. intro m. apply natsummationtimesdistl. rewrite <- D. apply idpath. Defined.

Lemma natsummationandfpszero (R : commrng) : forall m : nat, natsummation0 m (fun x : nat => fpszero R x) ~> (@rngunel1 R). Proof. intros R m. induction m. apply idpath. simpl. rewrite IHm. rewrite (rnglunax1 R). apply idpath. Defined.

Lemma ismonoidopfstimes (R : commrng) : ismonoidop (fpstimes R). Proof. intros. unfold ismonoidop. split. apply isassocfpstimes. split with (fpsone R). split. intro s. unfold fpstimes. change ((fun n : nat => natsummation0 n (fun x : nat => fpsone R x * s (minus n x))) ~> s). apply funextfun. intro n. destruct n. simpl. rewrite (rnglunax2 R). apply idpath. rewrite natsummationshift0. rewrite (rnglunax2 R). rewrite minus0r. assert (natsummation0 n (fun x : nat => fpsone R (S x) * s (minus n x)) ~> ((natsummation0 n (fun x : nat => fpszero R x)))) as f. apply natsummationpathsupperfixed. intros m m'. rewrite (rngmult0x R). apply idpath. change (natsummation0 n (fun x : nat => fpsone R (S x) * s (minus n x)) + s (S n) ~> (s (S n))). rewrite f. rewrite natsummationandfpszero. apply (rnglunax1 R).

intros s. unfold fpstimes. change ((fun n : nat => natsummation0 n (fun x : nat => s x * fpsone R (minus n x))) ~> s). apply funextfun. intro n. destruct n. simpl. rewrite (rngrunax2 R). apply idpath. change (natsummation0 n (fun x : nat => s x * fpsone R (minus (S n) x)) + s (S n) * fpsone R (minus n n) ~> s (S n)). rewrite minusnn0. rewrite (rngrunax2 R). assert (natsummation0 n (fun x : nat => s x * fpsone R (minus (S n) x)) ~> ((natsummation0 n (fun x : nat => fpszero R x)))) as f. apply natsummationpathsupperfixed. intros m m'. rewrite <- pathssminus. rewrite (rngmultx0 R). apply idpath. apply (natllehlthtrans _ n). assumption. apply natlthnsn. rewrite f. rewrite natsummationandfpszero. apply (rnglunax1 R). Defined.

Lemma iscommfpstimes (R : commrng) (s t : fps R) : fpstimes R s t ~> fpstimes R t s. Proof. intros. unfold fpstimes. change ((fun n : nat => natsummation0 n (fun x : nat => s x * t (minus n x))) ~> (fun n : nat => natsummation0 n (fun x : nat => t x * s (minus n x)))). apply funextfun. intro n.

assert (natsummation0 n (fun x : nat => s x * t (minus n x)) ~> (natsummation0 n (fun x : nat => t (minus n x) * s x))) as a0. apply maponpaths. apply funextfun. intro m. apply R. assert ((natsummation0 n (fun x : nat => t (minus n x) * s x)) ~> (natsummation0 n (funcomp (nattruncreverse n) (fun x : nat => t x * s (minus n x))))) as a1.

apply natsummationpathsupperfixed. intros m q. unfold funcomp. unfold nattruncreverse. destruct (natgthorleh m n). assert empty. apply isirreflnatlth with (n := n). apply natllehtrans with (m := m). apply h. assumption. contradiction. apply maponpaths. apply maponpaths. apply pathsinv0. apply doubleminuslehp. assumption. assert ((natsummation0 n (funcomp (nattruncreverse n) (fun x : nat => t x * s (minus n x)))) ~> natsummation0 n (fun x : nat => t x * s (minus n x))) as a2. apply pathsinv0. apply natsummationreindexing. apply

```
nattruncreverseisnattruncauto. exact ( pathscomp0 a0 ( pathscomp0 a1
a2 ) ). Defined.
```

```
Lemma isldistrfps ( R : commrng ) ( s t u : fps R ) : fpstimes R s (
fpsplus R t u ) ~> ( fpsplus R ( fpstimes R s t ) ( fpstimes R s u )
). Proof. intros. unfold fpstimes. unfold fpsplus. change ((fun n :
nat => natsummation0 n (fun x : nat => s x * (t (minus n x) + u (
minus n x)))) ~> (fun n : nat => natsummation0 n (fun x : nat => s x *
t (minus n x) + natsummation0 n (fun x : nat => s x * u (minus n x))))
). apply funextfun. intro upper. assert ( natsummation0 upper ( fun
x : nat => s x * ( t ( minus upper x ) + u ( minus upper x ) ) ) ~> (
natsummation0 upper ( fun x : nat => ( ( s x * t ( minus upper x ) ) +
( s x * u ( minus upper x ) ) ) ) ) ) as a0. apply maponpaths. apply
funextfun. intro n. apply R. assert ( ( natsummation0 upper ( fun x :
nat => ( ( s x * t ( minus upper x ) ) + ( s x * u ( minus upper x ) )
) ) ) ~> ( ( natsummation0 upper ( fun x : nat => s x * t ( minus
upper x ) ) ) + ( natsummation0 upper ( fun x : nat => s x * u ( minus
upper x ) ) ) ) ) as a1. apply natsummationplusdistr. exact (
pathscomp0 a0 a1 ). Defined.
```

```
Lemma isrdistrfps ( R : commrng ) ( s t u : fps R ) : fpstimes R (
fpsplus R t u ) s ~> ( fpsplus R ( fpstimes R t s ) ( fpstimes R u s )
). Proof. intros. unfold fpstimes. unfold fpsplus. change ((fun n :
nat => natsummation0 n (fun x : nat => (t x + u x) * s (minus n x)
)) ~> (fun n : nat => natsummation0 n (fun x : nat => t x * s (minus
n x) + natsummation0 n (fun x : nat => u x * s (minus n x)))) ).
apply funextfun. intro upper. assert ( natsummation0 upper ( fun x :
nat => ( t x + u x ) * s ( minus upper x ) ) ~> ( natsummation0 upper
( fun x : nat => ( ( t x * s ( minus upper x ) ) + ( u x * s ( minus
upper x ) ) ) ) ) ) as a0. apply maponpaths. apply funextfun. intro
n. apply R. assert ( ( natsummation0 upper ( fun x : nat => ( ( t x *
s ( minus upper x ) ) + ( u x * s ( minus upper x ) ) ) ) ) ~> ( (
natsummation0 upper ( fun x : nat => t x * s ( minus upper x ) ) ) + (
natsummation0 upper ( fun x : nat => u x * s ( minus upper x ) ) ) ) )
as a1. apply natsummationplusdistr. exact ( pathscomp0 a0 a1 ).
Defined.
```

```
Definition fpsrng ( R : commrng ) := setwith2binoppair ( hSetpair (
secong R ) ( isasetfps R ) ) ( dirprodpair ( fpsplus R ) ( fpstimes R
) ).
```

```
Theorem fpsiscommrng ( R : commrng ) : iscommrng ( fpsrng R ). Proof.
intro. unfold iscommrng. unfold iscommrngops. split. unfold
isrngops. split. split. unfold isabgrp. split. exact ( isgropfpsplus
R ). exact ( iscommfpsplus R ). exact ( ismonoidopfpstimes R ).
unfold isdistr. split. unfold isdistr. intros. apply ( isldistrfps R
). unfold isrdistr. intros. apply ( isrdistrfps R ). unfold
iscomm. intros. apply ( iscommfpstimes R ). Defined.
```

```
Definition fpscommrng ( R : commrng ) : commrng := commrngpair (
fpsrng R ) ( fpsiscommrng R ).
```

```
Definition fpsshift { R : commrng } ( a : fpscommrng R ) : fpscommrng
R := fun n : nat => a ( S n ).
```

```
Lemma fpsshiftandmult { R : commrng } ( a b : fpscommrng R ) ( p : b
0%nat ~> 0 ) : forall n : nat, ( a * b ) ( S n ) ~> ( ( a * ( fpsshift
b ) ) n ). Proof. intros. induction n. change ( a * b ) with (
fpstimes R a b ). change ( a * fpsshift b ) with ( fpstimes R a (
fpsshift b ) ). unfold fpstimes. unfold fpsshift. simpl. rewrite
p. rewrite ( rngmultx0 R ). rewrite ( rngrunax1 R ). apply idpath.
change ( a * b ) with ( fpstimes R a b ). change ( a * fpsshift b )
with ( fpstimes R a ( fpsshift b ) ). unfold fpsshift. unfold
fpstimes. change ( natsummation0 ( S ( S n ) ) ( fun x : nat => a x * b
(minus ( S ( S n ) x ) ) ) with ( ( natsummation0 ( S n ) ( fun x : nat
=> a x * b ( minus ( S ( S n ) x ) ) ) + a ( S ( S n ) ) * b ( minus
```

```
( S ( S n ) ) ( S ( S n ) ) ) ). rewrite minusn0. rewrite p. rewrite
( rngmultx0 R ). rewrite rngrunax1. apply
natsummationpathsupperfixed. intros x j. apply maponpaths. apply
maponpaths. rewrite pathssminus. apply idpath. apply ( natlehlthtrans
_ ( S n ) _ ). assumption. apply natlthnsn. Defined.
```

(** * IV. Apartness relation on formal power series *)

```
Lemma apartbinarysum0 ( R : acommrng ) ( a b : R ) ( p : a + b # 0 ) :
hdisj ( a # 0 ) ( b # 0 ). Proof. intros. intros P s. apply (
acommrng_acotrans R ( a + b ) a 0 p ). intro k. destruct k as [ l | r
]. apply s. apply ii2. assert ( a + b # a ) as l'. apply l. assert (
( a + b ) # ( a + 0 ) ) as l''. rewrite rngrunax1. assumption. apply
( ( pr1 ( acommrng_aadd R ) ) a b 0 ). assumption. apply s. apply
iii. assumption. Defined.
```

```
Lemma apartnatsummation0 ( R : acommrng ) ( upper : nat ) ( f : nat ->
R ) ( p : ( natsummation0 upper f ) # 0 ) : hexists ( fun n : nat =>
dirprod ( natleh n upper ) ( f n # 0 ) ). Proof. intros R
upper. induction upper. simpl. intros. intros P s. apply s. split with
0%nat. split. intros g. simpl in g. apply
nopathsfalsestotru. assumption. assumption. intros. intros P s. simpl
in p. apply ( apartbinarysum0 R _ _ p ). intro k. destruct k as [ l |
r ]. apply ( lHupper f l ). intro k. destruct k as [ n ab ]. destruct
ab as [ a b ]. apply s. split with n. split. apply ( istransnatleh _
upper _ ). assumption. apply natlthtoleh. apply
natlthnsn. assumption. apply s. split with ( S upper ). split. apply
isrefinatleh. assumption. Defined.
```

```
Definition fpsapart0 ( R : acommrng ) : hrel ( fpscommrng R ) := fun s
t : fpscommrng R => ( hexists ( fun n : nat => ( s n # t n ) ) ).
```

```
Definition fpsapart ( R : acommrng ) : apart ( fpscommrng R ). Proof.
intros. split with ( fpsapart0 R ). split. intros s f. assert (
hfalse ) as i. apply f. intro k. destruct k as [ n p ]. apply (
acommrng_airrefl R ( s n ) p ). apply i. split. intros s t p
j. apply p. intro k. destruct k as [ n q ]. apply j. split with
n. apply ( acommrng_asymp R ( s n ) ( t n ) q ). intros s t u p
j. apply p. intro k. destruct k as [ n q ]. apply ( acommrng_acotrans
R ( s n ) ( t n ) ( u n ) q ). intro l. destruct l as [ l | r ].
apply j. apply iii. intros v V. apply V. split with n. assumption.
apply j. apply ii2. intros v V. apply V. split with n. assumption.
Defined.
```

```
Lemma fpsapartisbinopapartplusl ( R : acommrng ) : isbinopapartl (
fpsapart R ) ( @opl ( fpscommrng R ) ). Proof. intros. intros a b c
p. intros P s. apply p. intro k. destruct k as [ n q ]. apply
s. change ( ( a + b ) n ) with ( ( a n ) + ( b n ) ) in q. change ( (
a + c ) n ) with ( ( a n ) + ( c n ) ) in q. split with n. apply ( (
pr1 ( acommrng_aadd R ) ) ( a n ) ( b n ) ( c n ) q ). Defined.
```

```
Lemma fpsapartisbinopapartplusr ( R : acommrng ) : isbinopapartl (
fpsapart R ) ( @opl ( fpscommrng R ) ). Proof. intros. intros a b c
p. rewrite ( rngcomm1 ( fpscommrng R ) ) in p. rewrite ( rngcomm1 (
fpscommrng R ) c ) in p. apply ( fpsapartisbinopapartplusl _ a b c
). assumption. Defined.
```

```
Lemma fpsapartisbinopapartmultl ( R : acommrng ) : isbinopapartl (
fpsapart R ) ( @op2 ( fpsrng R ) ). Proof. intros. intros a b c
p. intros P s. apply p. intro k. destruct k as [ n q ]. change ( ( a
* b ) n ) with ( natsummation0 n ( fun x : nat => ( a x ) * ( b (
minus n x ) ) ) ) in q. change ( ( a * c ) n ) with ( natsummation0 n
( fun x : nat => ( a x ) * ( c ( minus n x ) ) ) ) in q. assert (
natsummation0 n ( fun x : nat => ( a x * b ( minus n x ) ) - ( a x * c (
minus n x ) ) ) # 0 ) as q'. assert ( natsummation0 n ( fun x : nat
=> ( a x * b ( minus n x ) ) ) - natsummation0 n ( fun x : nat => ( a
```

```
x * c ( minus n x ) ) ) # 0 ) as q''. apply aaminuszero. assumption.
assert ( ( fun x : nat => a x * b (minus n x) - a x * c ( minus n x) )
  ~> ( fun x : nat => a x * b ( minus n x) + ( - 1%rng ) * ( a x * c (
    minus n x) ) ) ) as i. apply funextfun. intro x. apply
  maponpaths. rewrite <- ( rngmultwithminus1 R ). apply idpath. rewrite
  i. rewrite natsummationplusdistr. rewrite <- ( natsummationtimesdistr
    n ( fun x : nat => a x * c (minus n x) ) ( - 1%rng ) ). rewrite (
    rngmultwithminus1 R ). assumption. apply ( apartnatsummation0 R n _
    q' ). intro k. destruct k as [ m g ]. destruct g as [ g g' ]. apply
    s. split with ( minus n m ). apply ( ( pr1 ( acommrng_amult R ) ) ( a
    m ) ( b ( minus n m ) ) ( c ( minus n m ) ) ). apply
    aminuszeroa. assumption. Defined.
```

```
Lemma fpsapartisbinopapartmultr ( R : acommrng ) : isbinopaparttr (
  fpsapart R ) ( @op2 ( fperng R ) ). Proof. intros. intros a b c
  p. rewrite ( rngcomm2 ( fpscommrng R ) ) in p. rewrite ( rngcomm2 (
    fpscommrng R ) c ) in p. apply ( fpsapartisbinopapartmultl _ a b c
    ). assumption. Defined.
```

```
Definition acommrngfps ( R : acommrng ) : acommrng. Proof.
  intros. split with ( fpscommrng R ). split with ( fpsapart R
  ). split. split. apply ( fpsapartisbinopapartplusr R ). apply (
    fpsapartisbinopapartplusr R ). split. apply (
    fpsapartisbinopapartmultl R ). apply ( fpsapartisbinopapartmultr R ).
  Defined.
```

```
Definition isacommrngapartdec ( R : acommrng ) := isapartdec ( ( pr1 (
  pr2 R ) ) ).
```

```
Lemma leadingcoefficientapartdec ( R : aintdom ) ( a : fpscommrng R )
  ( is : isacommrngapartdec R ) ( p : a 0%nat # 0 ) : forall n : nat,
  forall b : fpscommrng R, ( b n # 0 ) -> ( ( acommrngapartrel (
    acommrngfps R ) ) ( a * b ) 0 ). Proof. intros R a is p n. induction
  n. intros b q. intros P s. apply s. split with 0%nat. change ( ( a *
    b ) 0%nat ) with ( ( a 0%nat ) * ( b 0%nat ) ). apply
```

```
R. assumption. assumption. intros b q. destruct ( is ( b 0%nat ) 0 )
  as [ left | right ]. intros P s. apply s. split with 0%nat. change (
  ( a * b ) 0%nat ) with ( ( a 0%nat ) * ( b 0%nat ) ). apply
  R. assumption. assumption. assert ( ( acommrngapartrel ( acommrngfps
  R ) ) ( a * ( fpsshift b ) ) 0 ) as j. apply IHn. unfold
  fpsshift. assumption. apply j. intro k. destruct k as [ k i ]. intros
  P s. apply s. rewrite <- ( fpsshiftandmult a b right k ) in i. split
  with ( S k ). assumption. Defined.
```

```
Lemma apartdecintdom0 ( R : aintdom ) ( is : isacommrngapartdec R ) :
  forall n : nat, forall a b : fpscommrng R, ( a n # 0 ) -> (
    acommrngapartrel ( acommrngfps R ) b 0 ) -> ( acommrngapartrel (
    acommrngfps R ) ( a * b ) 0 ). Proof. intros R is n. induction
  n. intros a b p q. apply q. intros k. destruct k as [ k k0 ]. apply (
    leadingcoefficientapartdec R a is p k ). assumption. intros a b p
  q. destruct ( is ( a 0%nat ) 0 ) as [ left | right ]. apply q. intros
  k. destruct k as [ k k0 ]. apply ( leadingcoefficientapartdec R a is
  left k ). assumption. assert ( acommrngapartrel ( acommrngfps R ) ( (
  fpsshift a ) * b ) 0 ) as i. apply IHn. unfold
  fpsshift. assumption. assumption. apply i. intros k. destruct k as [
  k k0 ]. intros P s. apply s. split with ( S k ). rewrite
  rngcomm2. rewrite fpsshiftandmult. rewrite
  rngcomm2. assumption. assumption. Defined.
```

```
Theorem apartdectoisaaintdomfps ( R : aintdom ) ( is :
  isacommrngapartdec R ) : aintdom. Proof. intros R. split with (
  acommrngfps R ). split. intros P s. apply s. split with 0%nat. change
  ( ( @rngunell1 ( fpscommrng R ) ) 0%nat ) with ( @rngunell1 R ). change
  ( @rngunell2 R # ( @rngunell1 R ) ). apply R. intros a b p q. apply
  p. intro n. destruct n as [ n n0 ]. apply ( apartdecintdom0 R is n )
  . assumption. assumption. Defined.
```

```
Close Scope rng_scope.
```

```
(** END OF FILE *)
```

7.5 The file frac.v

```
(** *The Heyting field of fractions for an apartness domain *)
```

```
(** By Alvaro Pelayo, Vladimir Voevodsky and Michael A. Warren *)
```

```
(** February 2011 and August 2012 *)
```

```
(** Settings *)
```

```
Add Rec LoadPath "../Generalities". Add Rec LoadPath "../hlevel1".
Add Rec LoadPath "../hlevel2". Add Rec LoadPath "../Algebra".
```

```
Unset Automatic Introduction. (** This line has to be removed for the
file to compile with Coq8.2 *)
```

```
(** Imports *)
```

```
Require Export lemmas.
```

```
(** * I. The field of fractions for an integrable domain with an
apartness relation *)
```

```
Open Scope rng_scope.
```

```
Section aint.
```

```
Variable A : aintdom.
```

```
Ltac permute := solve [ repeat rewrite rngassoc2; match goal with | [
  |- ?X ~> ?X ] => apply idpath | [ |- ?X * ?Y ~> ?X * ?Z ] => apply
  maponpaths; permute | [ |- ?Y * ?X ~> ?Z * ?X ] => apply (
  maponpaths ( fun x => x * X ) ); permute | [ |- ?X * ?Y ~> ?Y * ?X ]
=> apply rngcomm2 | [ |- ?X * ?Y ~> ?K ] => solve [ repeat rewrite
  <- rngassoc2; match goal with | [ |- ?H ~> ?Y * X ] => rewrite (
  @rngcomm2 A V X ); repeat rewrite rngassoc2; apply maponpaths;
  permute end | repeat rewrite rngassoc2; match goal with | [ |- ?H ~>
  ?Z * ?V ] => repeat rewrite <- rngassoc2; match goal with | [ |- ?W
  * Z ~> ?L ] => rewrite ( @rngcomm2 A W Z ); repeat rewrite
  rngassoc2; apply maponpaths; permute end end ] || [ |- ?X * ( ?Y * ?Z
  ) ~> ?K ] => rewrite ( @rngcomm2 A Y Z ); permute end | repeat
  rewrite <- rngassoc2; match goal with | [ |- ?X * ?Y ~> ?X * ?Z ] =>
  apply maponpaths; permute | [ |- ?Y * ?X ~> ?Z * ?X ] => apply (
  maponpaths ( fun x => x * X ) ); permute | [ |- ?X * ?Y ~> ?Y * ?X ]
=> apply rngcomm2 end | apply idpath | idtac "The tactic permute
does not apply to the current goal!" ].
```

```
Lemma azerorelcomp ( cd : dirprod A ( aintdomazerosubmonoid A ) ) ( ef
```

```

: dirprod A ( aintdomazerosubmonoid A ) ) ( p : ( pr1 cd ) * ( pr1 (
pr2 ef ) ) ^> ( ( pr1 ef ) * ( pr1 ( pr2 cd ) ) ) ) ( q : ( pr1 cd ) #
0 ) : ( pr1 ef ) # 0. Proof. intros. change ( ( @op2 A ( pr1 cd ) )
pr1 ( pr2 ef ) ) ^> ( @op2 A ( pr1 ef ) ( pr1 ( pr2 cd ) ) ) in p.
assert ( ( @op2 A ( pr1 cd ) ( pr1 ( pr2 ef ) ) ) # 0 ) as v. apply
A. assumption. apply ( pr2 ( pr2 ef ) ). rewrite p in v. apply ( pr1 (
timesazero v ) ). Defined.

```

```

Lemma azerolmultcomp { a b c : A } ( p : a # 0 ) ( q : b # c ) : a * b
# a * c. Proof. intros. apply aminuszerao. rewrite <-
rngminusdistr. apply ( pr2 A ). assumption. apply
aminuszerao. assumption. Defined.

```

```

Lemma azerormultcomp { a b c : A } ( p : a # 0 ) ( q : b # c ) : b * a
# c * a. Proof. intros. rewrite ( @rngcomm2 A b ). rewrite (
@rngcomm2 A c ). apply ( azerolmultcomp p q ). Defined.

```

```

Definition aflldfracapartrelpre : hrel ( dirprod A (
aintdomazerosubmonoid A ) ) := fun ab cd : _ => ( ( pr1 ab ) * ( pr1 (
pr2 cd ) ) ) # ( ( pr1 cd ) * ( pr1 ( pr2 ab ) ) ).

```

```

Lemma aflldfracapartiscomprel : iscomprelrel ( eqrelcommrngfrac A (
aintdomazerosubmonoid A ) ) ( aflldfracapartrelpre ). Proof. intros
ab cd ef gh p q. unfold aflldfracapartrelpre. destruct ab as [ a b
]. destruct b as [ b b' ]. destruct cd as [ c d ]. destruct d as [ d
d' ]. destruct ef as [ e f ]. destruct f as [ f f' ]. destruct gh as
[ g h ]. destruct h as [ h h' ]. simpl in *.

```

```

apply uahp. intro u. apply p. intro p'. apply q. intro q'. destruct
p' as [ p' j ]. destruct p' as [ i p' ]. destruct q' as [ q' j'
]. destruct q' as [ i' q' ]. simpl in *.

```

```

assert ( a * f * d * i * h * i' # e * b * d * i * h * i' ) as v0.
assert ( a * f * d # e * b * d ) as v0. apply azerormultcomp. apply
d'. assumption. assert ( a * f * d * i # e * b * d * i ) as
v1. apply azerormultcomp. assumption. assumption. assert ( a * f *
d * i * h # e * b * d * i * h ) as v2. apply azerormultcomp. apply
h'. assumption. apply azerormultcomp. assumption. assumption.
apply ( pr2 ( acommrng_amult A ) b ). apply ( pr2 ( acommrng_amult A
) f ). apply ( pr2 ( acommrng_amult A ) i ). apply ( pr2 (
acommrng_amult A ) i' ).

```

```

assert ( a * f * d * i * h * i' ^> c * h * b * f * i * i' ) as l.
assert ( a * f * d * i * h * i' ^> a * d * i * f * h * i' ) as l0.
change ( @op2 A ( @op2 A ( @op2 A ( @op2 A ( @op2 A a f ) d ) i ) h
) i' ^> @op2 A ( @op2 A ( @op2 A ( @op2 A ( @op2 A a d ) i ) f ) h )
i' ). permute. rewrite l0. rewrite j. change ( @op2 A ( @op2 A (
@op2 A ( @op2 A ( @op2 A c b ) i ) f ) h ) i' ^> @op2 A ( @op2 A (
@op2 A ( @op2 A ( @op2 A c h ) b ) f ) i ) i' ). permute. rewrite l
in v0. assert ( e * b * d * i * h * i' ^> g * d * b * f * i * i' )
as k. assert ( @op2 A ( @op2 A ( @op2 A ( @op2 A ( @op2 A e b ) d )
i ) h ) i' ^> @op2 A ( @op2 A ( @op2 A ( @op2 A ( @op2 A e h ) i' )
i ) b ) d ) as k0. permute. change ( @op2 A ( @op2 A ( @op2 A (
@op2 A ( @op2 A e b ) d ) i ) h ) i' ^> @op2 A ( @op2 A ( @op2 A (
@op2 A ( @op2 A g d ) b ) f ) i ) i' ). rewrite k0. assert ( @op2 A
( @op2 A e h ) i' ^> @op2 A ( @op2 A g f ) i' ) as j''. assumption.
rewrite j''. permute. rewrite k in v0. assumption.

```

```

intro u. apply p. intro p'. apply q. intro q'. destruct p' as [ p'
j ]. destruct p' as [ i p' ]. destruct q' as [ q' j' ]. destruct q'
as [ i' q' ]. simpl in *.

```

```

assert ( c * h * b * f * i * i' # g * d * b * f * i * i' ) as v.
apply azerormultcomp. apply q'. apply azerormultcomp. apply p'.
apply azerormultcomp. apply f'. apply azerormultcomp. apply
b'. assumption. apply ( pr2 ( acommrng_amult A ) d ). apply ( pr2 (

```

```

acommrng_amult A ) h ). apply ( pr2 ( acommrng_amult A ) i ). apply
( pr2 ( acommrng_amult A ) i' ).

```

```

assert ( c * h * b * f * i * i' ^> a * f * d * h * i * i' ) as k.
assert ( c * h * b * f * i * i' ^> c * b * i * f * h * i' ) as k0.
change ( @op2 A ( @op2 A ( @op2 A ( @op2 A ( @op2 A c h ) b ) f ) i
) i' ^> @op2 A ( @op2 A ( @op2 A ( @op2 A ( @op2 A c b ) i ) f ) h )
i' ). permute. rewrite k0. rewrite <- j. change ( @op2 A ( @op2 A (
@op2 A ( @op2 A ( @op2 A a d ) i ) f ) h ) i' ^> @op2 A ( @op2 A (
@op2 A ( @op2 A ( @op2 A a f ) d ) h ) i ) i' ). permute. rewrite k
in v. assert ( g * d * b * f * i * i' ^> e * b * d * h * i * i' )
as l. assert ( g * d * b * f * i * i' ^> g * f * i' * d * i * b )
as l0. change ( @op2 A ( @op2 A ( @op2 A ( @op2 A ( @op2 A g d ) b
) f ) i ) i' ^> @op2 A ( @op2 A ( @op2 A ( @op2 A ( @op2 A g f ) i'
) d ) i ) b ). permute. rewrite l0. rewrite <- j'. change ( @op2 A (
@op2 A ( @op2 A ( @op2 A ( @op2 A e h ) i' ) d ) i ) b ^> @op2 A (
@op2 A ( @op2 A ( @op2 A ( @op2 A e b ) d ) h ) i ) i'
). permute. rewrite l in v. assumption. Defined.

```

(** We now arrive at the apartness relation on the field of fractions
itself.*)

```

Definition aflldfracapartrel := quotrel aflldfracapartiscomprel.

```

```

Lemma isirreflafdfracapartrelpre : isirrefl aflldfracapartrelpre.
Proof. intros ab. apply acommrng_airrefl. Defined.

```

```

Lemma issymmaflldfracapartrelpre : issymm aflldfracapartrelpre. Proof.
intros ab cd. apply ( acommrng_asymm A ). Defined.

```

```

Lemma iscotransaflldfracapartrelpre : iscotrans aflldfracapartrelpre.
Proof. intros ab cd ef p. destruct ab as [ a b ]. destruct b as [ b
b' ]. destruct cd as [ c d ]. destruct d as [ d d' ]. destruct ef as
[ e f ]. destruct f as [ f f' ]. assert ( a * f * d # e * b * d ) as
v. apply azerormultcomp. assumption. assumption. apply ( (
acommrng_acotrans A ( a * f * d ) ( c * b * f ) ( e * b * d ) ) v
). intro u. intros P k. apply k. unfold aflldfracapartrelpre in
*. simpl in *. destruct u as [ left | right ]. apply ii1. apply ( pr2
( acommrng_amult A ) f ). assert ( @op2 A ( @op2 A a f ) d ^> @op2 A
( @op2 A a d ) f ) as i. permute. change ( @op2 A ( @op2 A a d ) f #
@op2 A ( @op2 A c b ) f ). rewrite <- i. assumption. apply ii2. apply
( pr2 ( acommrng_amult A ) b ). assert ( @op2 A ( @op2 A c f ) b ^>
@op2 A ( @op2 A c b ) f ) as i. permute. change ( @op2 A ( @op2 A c f
) b # @op2 A ( @op2 A e d ) b ). rewrite i. assert ( @op2 A ( @op2 A
e d ) b ^> @op2 A ( @op2 A e b ) d ) as j. permute. change ( @op2 A (
@op2 A c b ) f # @op2 A ( @op2 A e d ) b ). rewrite j. assumption.
Defined.

```

```

Lemma isapartaflldfracapartrel : isapart aflldfracapartrel. Proof.
intros. split. apply isirreflquotrel. exact (
isirreflafdfracapartrelpre ). split. apply issymmquotrel. exact (
issymmaflldfracapartrelpre ). apply iscotransquotrel. exact (
iscotransaflldfracapartrelpre ). Defined.

```

```

Definition aflldfracapart : apart ( commrngfrac A
( aintdomazerosubmonoid A ) ). Proof. intros. unfold apart. split with
aflldfracapartrel. exact isapartaflldfracapartrel. Defined.

```

```

Lemma isbinapartlafdfracop1 : isbinopapartl aflldfracapart op1.
Proof. intros. unfold isbinopapartl. assert ( forall a b c :
commrngfrac A ( aintdomazerosubmonoid A ), isaprop ( pr1
(aflldfracapart) ( commrngfracop1 A ( aintdomazerosubmonoid A ) a b ) (
commrngfracop1 A ( aintdomazerosubmonoid A ) a c ) -> pr1
(aflldfracapart) b c ) ) as int. intros a b c. apply impred. intro
p. apply ( pr1 ( aflldfracapart ) b c ). apply ( setquotuniv3prop _ (
fun a b c => hProppair _ ( int a b c ) ) ). intros ab cd ef

```

```
p. destruct ab as [ a b ]. destruct b as [ b b' ]. destruct cd as [ c
d ], destruct d as [ d d' ]. destruct ef as [ e f ]. destruct f as [
f f' ]. unfold aflldfracapart in *. simpl. unfold
aflldfracapartrel. unfold quotrel. rewrite setquotuniv2comm. unfold
aflldfracapartrelpre. simpl.
```

```
assert ( aflldfracapartrelpre ( dirprodpair ( @op1 A ( @op2 A d a ) (
@op2 A b c ) ) ( @op ( aintdomazerosubmonoid A ) ( tpair b b' ) (
tpair d d' ) ) ) ( dirprodpair ( @op1 A ( @op2 A f a ) ( @op2 A b e
) ) ( @op ( aintdomazerosubmonoid A ) ( tpair b b' ) ( tpair f f' )
) ) ) as u. apply p. unfold aflldfracapartrelpre in u. simpl in
u. rewrite 2! ( @rngrdistr A ) in u. repeat rewrite <- rngassoc2 in
u. assert ( (@op2 (pr1rng (commrngtoring (acommrngtocommrng
(plaintdom A)))) (@op2 (pr1rng (commrngtoring (acommrngtocommrng
(plaintdom A)))) (@op2 (@pr1 setwith2binop (fun X : setwith2binop
=> @iscommrngops (prisetwith2binop X) (@op1 X) (@op2 X))
(acommrngtocommrng (plaintdom A))) d a) b) f) ^> (@op2 (pr1rng
(commrngtoring (acommrngtocommrng (plaintdom A)))) (@op2 (pr1rng
(commrngtoring (acommrngtocommrng (plaintdom A)))) (@op2 (@pr1
setwith2binop (fun X : setwith2binop => @iscommrngops
(prisetwith2binop X) (@op1 X) (@op2 X)) (acommrngtocommrng
(plaintdom A))) f a) b) d) ) as i. permute. rewrite i in u. assert
( (@op2 (pr1rng (commrngtoring (acommrngtocommrng (plaintdom A))))
(@op2 (pr1rng (commrngtoring (acommrngtocommrng (plaintdom A))))
(@op2 (@pr1 setwith2binop (fun X : setwith2binop => @iscommrngops
(prisetwith2binop X) (@op1 X) (@op2 X)) (acommrngtocommrng
(plaintdom A))) b c) b) f) ^> (@op2 (pr1rng (commrngtoring
(acommrngtocommrng (plaintdom A)))) (@op2 (pr1rng (commrngtoring
(acommrngtocommrng (plaintdom A)))) (@op2 (@pr1 setwith2binop (fun
X : setwith2binop => @iscommrngops (prisetwith2binop X) (@op1 X)
(@op2 X)) (acommrngtocommrng (plaintdom A))) c f) b) b) ) as
j. permute. rewrite j in u. assert ( (@op2 (pr1rng (commrngtoring
(acommrngtocommrng (plaintdom A)))) (@op2 (pr1rng (commrngtoring
(acommrngtocommrng (plaintdom A)))) (@op2 (@pr1 setwith2binop (fun
X : setwith2binop => @iscommrngops (prisetwith2binop X) (@op1 X)
(@op2 X)) (acommrngtocommrng (plaintdom A))) b e) b) d) ^> (@op2
(pr1rng (commrngtoring (acommrngtocommrng (plaintdom A)))) (@op2
(pr1rng (commrngtoring (acommrngtocommrng (plaintdom A)))) (@op2
(@pr1 setwith2binop (fun X : setwith2binop => @iscommrngops
(prisetwith2binop X) (@op1 X) (@op2 X)) (acommrngtocommrng
(plaintdom A))) e d) b) b) ) as j'. permute. rewrite j' in u.
apply ( pr2 ( acommrng_amult A ) b ). apply ( pr2 ( acommrng_amult
A ) b ). apply ( pr1 ( acommrng_aadd A ) ( f * a * b * d )
). assumption. Defined.
```

```
Lemma isbinapartraflldfracop1 : isbinopapart aflldfracapart op1.
Proof. intros a b c. rewrite ( rngcomm1 ). rewrite ( rngcomm1 _ c
). apply isbinapartlaflldfracop1. Defined.
```

```
Lemma isbinapartlaflldfracop2 : isbinopapart1 aflldfracapart op2.
Proof. intros. unfold isbinopapart1. assert ( forall a b c :
commrngfrac A ( aintdomazerosubmonoid A ), isaprop ( pr1
(aflldfracapart ) ( commrngfracop2 A ( aintdomazerosubmonoid A ) a b) (
commrngfracop2 A ( aintdomazerosubmonoid A ) a c ) -> pr1
(aflldfracapart ) b c ) ) as int. intros a b c. apply impred. intro
p. apply ( pr1 ( aflldfracapart ) b c ). apply ( setquotuniv3prop _ (
fun a b c => hProppair _ ( int a b c ) ) ). intros ab cd ef p.
```

```
destruct ab as [ a b ]. destruct b as [ b b' ]. destruct cd as [ c
d ]. destruct d as [ d d' ]. destruct ef as [ e f ]. destruct f as
[ f f' ].
```

```
assert ( aflldfracapartrelpre ( dirprodpair ( ( a * c ) ) ( @op (
aintdomazerosubmonoid A ) ( tpair b b' ) ( tpair d d' ) ) ) (
dirprodpair ( a * e ) ( @op ( aintdomazerosubmonoid A ) ( tpair b b'
) ( tpair f f' ) ) ) ) as u. apply p. unfold aflldfracapart in
```

```
*. simpl. unfold aflldfracapartrel. unfold quotrel. rewrite (
setquotuniv2comm ( eqrelcommrngfrac A ( aintdomazerosubmonoid A ) )
). unfold aflldfracapartrelpre in *. simpl. simpl in u. apply ( pr2
( acommrng_amult A ) a ). apply ( pr2 ( acommrng_amult A ) b ).
```

```
assert ( c * f * a * b ^> (@op2 (@pr1 setwith2binop (fun X :
setwith2binop => @iscommrngops (prisetwith2binop X) (@op1 X) (@op2
X)) (acommrngtocommrng (plaintdom A))) (@op2 (@pr1 setwith2binop
(fun X : setwith2binop => @iscommrngops (prisetwith2binop X) (@op1
X) (@op2 X)) (acommrngtocommrng (plaintdom A))) a c) (@op2 (@pr1
setwith2binop (fun X : setwith2binop => @iscommrngops
(prisetwith2binop X) (@op1 X) (@op2 X)) (acommrngtocommrng
(plaintdom A))) b f) ) as i. change ( c * f * a * b ^> a * c * ( b
* f ) ). permute. change ( c * f * a * b # e * d * a * b ). rewrite
i. assert ( e * d * a * b ^> (@op2 (@pr1 setwith2binop (fun X :
setwith2binop => @iscommrngops (prisetwith2binop X) (@op1 X) (@op2
X)) (acommrngtocommrng (plaintdom A))) (@op2 (@pr1 setwith2binop
(fun X : setwith2binop => @iscommrngops (prisetwith2binop X) (@op1
X) (@op2 X)) (acommrngtocommrng (plaintdom A))) a e) (@op2 (@pr1
setwith2binop (fun X : setwith2binop => @iscommrngops
(prisetwith2binop X) (@op1 X) (@op2 X)) (acommrngtocommrng
(plaintdom A))) b d) ) as i'. change ( e * d * a * b ^> a * e * (
b * d ) ). permute. rewrite i'. assumption. Defined.
```

```
Lemma isbinapartraflldfracop2 : isbinopapart ( aflldfracapart ) op2.
Proof. intros a b c. rewrite rngcomm2. rewrite ( rngcomm2 _ c
). apply isbinapartlaflldfracop2. Defined.
```

```
Definition aflldfrac0 : acommrng. Proof. intros. split with (
commrngfrac A ( aintdomazerosubmonoid A ) ). split with (
aflldfracapart ). split. split. apply ( isbinapartlaflldfracop1
). apply ( isbinapartraflldfracop1 ). split. apply (
isbinapartlaflldfracop2 ). apply ( isbinapartraflldfracop2 ). Defined.
```

```
Definition aflldfracmultinvint ( ab : dirprod A ( aintdomazerosubmonoid
A ) ) ( is : aflldfracapartrelpre ab ( dirprodpair ( @rngunel1 A ) (
unel ( aintdomazerosubmonoid A ) ) ) ) : dirprod A (
aintdomazerosubmonoid A ). Proof. intros. destruct ab as [ a b
]. destruct b as [ b b' ]. split with b. simpl in is. split with
a. unfold aflldfracapartrelpre in is. simpl in is. change ( a # 0
). rewrite ( @rngmult0x A ) in is. rewrite ( @rngrunax2 A ) in
is. assumption. Defined.
```

```
Definition aflldfracmultinv ( a : aflldfrac0 ) ( is : a # 0 ) :
multinvpair aflldfrac0 a. Proof. intros. assert ( forall b :
aflldfrac0, isaprop ( b # 0 -> multinvpair aflldfrac0 b ) ) as int.
intros. apply impred. intro p. apply ( isapropmultinvpair aflldfrac0 ).
assert ( forall b : aflldfrac0, b # 0 -> multinvpair aflldfrac0 b ) as
p. apply ( setquotunivprop _ ( fun x0 => hProppair _ ( int x0 ) ) ).
intros bc q. destruct bc as [ b c ]. assert ( aflldfracapartrelpre (
dirprodpair b c ) ( dirprodpair ( @rngunel1 A ) ( unel (
aintdomazerosubmonoid A ) ) ) ) as is'. apply q. split with
(setquotpr (eqrelcommrngfrac A (aintdomazerosubmonoid A)) (
aflldfracmultinvint ( dirprodpair b c ) is' ) ).
```

```
split. change ( setquotpr ( eqrelcommrngfrac A (
aintdomazerosubmonoid A ) ) ( dirprodpair ( @op2 A ( pr1 (
aflldfracmultinvint ( dirprodpair b c ) is' ) ) b ) ( @op (
aintdomazerosubmonoid A ) ( pr2 ( aflldfracmultinvint ( dirprodpair b
c ) is' ) ) c ) ) ^> ( commrngfracunel2 A ( aintdomazerosubmonoid A
) ) ). apply iscompsetquotpr. unfold commrngfracunel2int. destruct
c as [ c c' ]. simpl. apply total2tohexists. split with (
carrierpair ( fun x : pr1 A => x # 0 ) 1 ( pr1 ( pr2 A ) ) ).
simpl. rewrite 3! ( @rngrunax2 A ). rewrite ( @rnglunax2 A ). apply
( @rngcomm2 A ).
```

```

change ( setquotpr ( eqrelcommrngfrac A ( aintdomazerosubmonoid A )
) ( dirprodpair ( @op2 A b ( pr1 ( aflldfracmultinvint ( dirprodpair b
c ) is' ) ) ) ( @op ( aintdomazerosubmonoid A ) c ( pr2 (
aflldfracmultinvint ( dirprodpair b c ) is' ) ) ) ) ) ^> (
commrngfracunel2 A ( aintdomazerosubmonoid A ) ) ). apply
iscompsetquotpr. destruct c as [ c c' ]. simpl. apply
total2tohexists. split with ( carrierpair ( fun x : pr1 A => x # 0 )
1 ( pr1 ( pr2 A ) ) ). simpl. rewrite 3! ( @rngrunax2 A ). rewrite (
@rnglunax2 A ). apply ( @rngcomm2 A ). apply p. assumption.
Defined.

```

```

Theorem aflldfracisafld : isaafld aflldfrac0. Proof. intros. split.
change ( ( aflldfracapartrel ) ( @rngunel2 ( commrngfrac A (

```

7.6 The file zmodp.v

```

(** *Integers mod p *)

```

```

(** By Alvaro Pelayo, Vladimir Voevodsky and Michael A. Warren *)

```

```

(** December 2011 *)

```

```

(** Settings *)

```

```

Add Rec LoadPath "../Generalities". Add Rec LoadPath "../hlevel1".
Add Rec LoadPath "../hlevel2". Add Rec LoadPath
"../Proof_of_Extensionality". Add Rec LoadPath "../Algebra".

```

```

Unset Automatic Introduction. (** This line has to be removed for the
file to compile with Coq8.2 *)

```

```

(** Imports *)

```

```

Require Export lemmas.

```

```

Open Scope hz_scope.

```

```

(** * I. Divisibility and the division algorithm *)

```

```

Definition hzdiv0 : hz -> hz -> hz -> UU := fun n m k => ( n * k ^> m
).

```

```

Definition hzdiv : hz -> hz -> hProp := fun n m => hexists ( fun k :
hz => hzdiv0 n m k ).

```

```

Lemma hzdivisrefl : isrefl ( hzdiv ). Proof. unfold
isrefl. intro. unfold hzdiv. apply total2tohexists. split with
1. apply hzmultr1. Defined.

```

```

Lemma hzdivistrans : istrans ( hzdiv ). Proof. intros a b c p
q. apply p. intro k. destruct k as [ k f ]. apply q. intro
l. destruct l as [ l g ]. intros P s. apply s. unfold hzdiv0 in f,g.
split with ( k * l ). unfold hzdiv0. rewrite <- hzmuiltassoc. rewrite
f. assumption. Defined.

```

```

Lemma hzdivlinearcombleft ( a b c d : hz ) ( f : a ^> ( b + c ) ) ( x
: hzdiv d a ) ( y : hzdiv d b ) : hzdiv d c. Proof. intros a b c d f
x y P s. apply x. intro x'. apply y. intro y'. destruct x' as [ k g
]. destruct y' as [ l h ]. unfold hzdiv0 in *. apply s. split with (
k + - l ). rewrite hzdistr. rewrite g. rewrite ( rngmultminus hz

```

```

aintdomazerosubmonoid A ) ) ) ( @rngunel1 ( commrngfrac A (
aintdomazerosubmonoid A ) ) ) ). unfold aflldfracapartrel. cut ( (
@op2 A ( @rngunel2 A ) ( @rngunel2 A ) ) # ( @op2 A ( @rngunel1 A ) (
@rngunel2 A ) ) ). intro v. apply v. rewrite 2! ( @rngrunax2 A
). apply A.

```

```

intros a p. apply aflldfracmultinv. assumption. Defined.

```

```

Definition aflldfrac := aflldpair aflldfrac0 aflldfracisafld.

```

```

End aint.

```

```

Close Scope rng_scope.
(** END OF FILE *)

```

```

). change ( ( a + ( - ( d * l ) ) ) %hz ^> c ). rewrite h. apply (
hzpluscan _ _ b ). rewrite hzplusassoc. rewrite hzlminus. rewrite
hzplusr0. rewrite hzpluscomm. assumption. Defined.

```

```

Lemma hzdivlinearcombright ( a b c d : hz ) ( f : a ^> ( b + c ) ) ( x
: hzdiv d b ) ( y : hzdiv d c ) : hzdiv d a. Proof. intros a b c d f
x y P s. apply x. intro x'. apply y. intro y'. destruct x' as [ k g
]. destruct y' as [ l h ]. unfold hzdiv0 in *. apply s. split with (
k + l ). rewrite hzdistr. change ( ( d * k + d * l ) %hz ^> a
). rewrite g, h, f. apply idpath. Defined.

```

```

Lemma divalgorithmnonneg ( n : nat ) ( m : nat ) ( p : hzlth 0 (
nattohz m ) ) : total2 ( fun qr : dirprod hz hz => ( ( dirprod (
nattohz n ^> ( ( ( nattohz m ) * ( pr1 qr ) ) + ( pr2 qr ) ) ) (
dirprod ( hzleh 0 ( pr2 qr ) ) ( hzlth ( pr2 qr ) ( nattohz ( m ) ) )
) ) ) ). Proof. intro. intro. induction n. intros. split with (
dirprodpair 0 0 ). split. simpl. rewrite ( rngrunax1 hz ). rewrite (
rngmultx0 hz ). rewrite nattohzand0. change ( 0 ^> 0 %hz ). apply
idpath. split. apply isreflhzleh. assumption.

```

```

intro p. set ( q' := pr1 ( pr1 ( IHn p ) ) ). set ( r' := pr2 (
pr1 ( IHn p ) ) ). set ( f := pr1 ( pr2 ( IHn p ) ) ). assert (
hzleh ( r' + 1 ) ( nattohz m ) ) as p'. assert ( hzlth ( r' + 1 )
( nattohz m + 1 ) ) as p''. apply hzlthandplusr. apply ( pr2 (
pr2 ( pr2 ( IHn p ) ) ) ). apply hzlthstoleh. assumption. set (
choice := hzlehchoice ( r' + 1 ) ( nattohz m ) p' ). destruct
choice as [ k | h ]. split with ( dirprodpair q' ( r' + 1 )
). split. rewrite ( nattohzandS _ ). rewrite hzpluscomm. rewrite
f. change ( nattohz m * q' + r' + 1 ^> ( nattohz m * q' + ( r' + 1
) ) ). apply rngassoc1. split. apply ( istranshzleh 0 r' ( r' + 1
) ). apply ( ( pr2 ( pr2 ( IHn p ) ) ) ). apply hzlthtoleh. apply
hzlthnsn. assumption. split with ( dirprodpair ( q' + 1 ) 0
). split. rewrite ( nattohzandS _ ). rewrite hzpluscomm. rewrite
f. change ( nattohz m * q' + r' + 1 ^> ( nattohz m * ( q' + 1 ) +
0 ) ). rewrite ( hzplusassoc ). rewrite h. rewrite ( rngldistr _
q' _ ). rewrite rngrunax2. rewrite hzplusr0. apply idpath.
split. apply isreflhzleh. assumption. Defined.

```

```

(* A test of the division algorithm for non-negative integers: Lemma
testlemmal : ( hzneq 0 ( 1 ) ). Proof. change 0 with ( nattohz 0 %nat
). rewrite <- nattohzand1. apply nattohzandneq. intro f. apply (
isirreflnatlth 1 ). assert ( natlth 0 1 ) as i. apply
natlthnsn. rewrite <- f in *. assumption. Defined.

```

```

Lemma testlemma2 : (hzneq 0 (1 + 1)). Proof. change 0 with (
  nattohz 0%nat). rewrite <- nattohzand1. rewrite <-
  nattohzandplus. apply nattohzandneq. assert (natneq (1 + 1) 0) as
  x. apply (natgthtoneq (1 + 1) 0). simpl. auto. intro f. apply
  x. apply pathsinv0. assumption. Defined.

```

```

Lemma testlemma21 : hzlth 0 (nattohz 2). Proof. change 0 with (
  nattohz 0%nat). apply nattohzandlth. apply (istransnatlth _ 1
  ). apply natlthnsn. apply natlthnsn. Defined.

```

```

Lemma testlemma3 : hzlth 0 (nattohz 3). Proof. apply (
  istranshzlth _ (nattohz 2)). apply testlemma21. change 0 with (
  nattohz 0%nat). apply nattohzandlth. apply natlthnsn. Defined.

```

```

Lemma testlemma9 : hzlth 0 (nattohz 9). Proof. apply (
  istranshzlth _ (nattohz 3)). apply testlemma3. apply (
  istranshzlth _ (nattohz 6)). apply testlemma3. apply testlemma3.
  Defined.

```

```

Eval lazy in hzabsval (pr1 (pr1 (divalgorithmonneg 1 (1 + 1)
  testlemma21))). Eval lazy in hzabsval (pr1 (pr1 (
  divalgorithmonneg (5) (1 + 1) testlemma21))). Eval lazy in
  hzabsval (pr2 (pr1 (divalgorithmonneg (5) (1 + 1) testlemma21
  ))). Eval lazy in hzabsval (pr1 (pr1 (divalgorithmonneg 16 3
  testlemma3))). Eval lazy in hzabsval (pr2 (pr1 (
  divalgorithmonneg 16 3 testlemma3))). Eval lazy in hzabsval (pr1
  (pr1 (divalgorithmonneg 18 9 testlemma9))). Eval lazy in
  hzabsval (pr2 (pr1 (divalgorithmonneg 18 9 testlemma9))). *)

```

```

Theorem divalgorithmeexists (n m : hz) (p : hzneq 0 m) : total2 (
  fun qr : dirprod hz hz => ((dirprod (n ^> ((m * (pr1 qr)) + (
  pr2 qr))) (dirprod (hzleh 0 (pr2 qr)) (hzlth (pr2 qr) (
  nattohz (hzabsval m))))))). Proof. intros. destruct (
  hzlthorgeh n 0) as [n_neg | n_nonneg]. destruct (hzlthorgeh m 0)
  as [m_neg | m_nonneg].

```

```

(*Case I: n<0, m<0:*) set (n' := hzabsval n). set (m' := hzabsval
  m). assert (nattohz m' ^> (- m)) as f. apply
  hzabsvalh0. assumption. assert (- n ^> - (nattohz n')) as
  f0. rewrite <- (hzabsvalh0 n_neg). rewrite (hzabsvalh0 n_neg
  ). unfold n'. rewrite (hzabsvalh0 n_neg). apply idpath. assert
  (hzlth 0 (nattohz m')) as p'. assert (hzlth 0 (- m)) as q.
  apply hzlh0andminus. assumption. rewrite f. assumption. set (a :=
  divalgorithmonneg n' m' p'). set (q := pr1 (pr1 a)). set (r
  := pr2 (pr1 a)). set (Q := q + 1). set (R := - m - r).

```

```

destruct (hzlehchoice 0 r (pr1 (pr2 (pr2 a)))) as [less |
  equal]. split with (dirprodpair Q R). split.

```

```

rewrite (pathsinv0 (rngminusminus hz n)). assert (- nattohz n'
  ^> (m * Q + R)) as f1. unfold Q. unfold R. rewrite (pr1 (
  pr2 a)). change (pr1 (pr1 a)) with q. change (pr2 (pr1
  a)) with r. rewrite hzaddinvplus. rewrite <- (rnglmultminus hz
  ). rewrite f. rewrite (rngminusminus ). rewrite (rngldistr _ _
  ). rewrite (hzmultl1). change ((m * q) + - r ^> ((m * q
  + m) + (- m - r))). rewrite (hzplusassoc). rewrite <- (
  hzplusassoc m _). change (m + - m) with (m - m). rewrite (
  hzrminus ). rewrite (hzplusl0). apply idpath. exact (
  pathscmp0 f0 f1). split. unfold R. assert (hzlth r (- m))
  as u. rewrite <- hzabsvalh0. apply (pr2 (pr2 (pr2 a)))
  ). apply hzlthtoleh. assumption. rewrite <- (hzlminus m). change
  (pr2 (dirprodpair Q (- m - r))) with (- m - r). apply
  hzlehandplusl. apply hzlthtoleh. rewrite <- (rngminusminus hz m
  ). apply hzlthminusswap. assumption. unfold R. unfold m'. rewrite
  hzabsvalh0. change (hzlth (- m + - r) (- m)). assert (
  hzlth (- m - r) (- m + 0)) as u. apply hzlthandplusl. apply

```

```

hzgth0andminus. apply less. assert (- m + 0 ^> (- m)) as f'.
  apply hzplusr0. exact (transportf (fun x : _ => hzlth (- m + -
  r) x) f' u). apply hzlthtoleh. assumption. split with
  (dirprodpair q 0). split. rewrite <- (rngminusminus hz n).
  assert (- nattohz n' ^> (m * q + 0)) as f1. rewrite (pr1 (
  pr2 (a))). change (pr1 (pr1 a)) with q. change (pr2 (pr1
  a)) with r. rewrite hzplusr0. rewrite (pathsinv0 equal
  ). rewrite (hzplusr0). assert (- (nattohz m' * q) ^> ((- (
  nattohz m')) * q)) as f2. apply pathsinv0. apply
  rnglmultminus. rewrite f2. unfold m'. rewrite hzabsvalh0. apply
  (maponpaths (fun x : _ => x * q)). apply rngminusminus. apply
  hzlthtoleh. assumption. exact (pathscmp0 f0 f1). split. change
  (pr2 (dirprodpair q 0)) with 0. apply (isreflhzleh). rewrite
  equal. change (pr2 (dirprodpair q r)) with r. apply (pr2 (
  pr2 (pr2 (a)))).

```

```

destruct (hzgehchoice m 0 m_nonneg) as [h | k].

```

```

(*****

```

```

(*Case II: n<0, m>0. *)

```

```

assert (hzlth 0 (nattohz (hzabsval m))) as p'. rewrite (
  hzabsvalgth0). apply h. assumption. set (a :=
  divalgorithmonneg (hzabsval n) (hzabsval m) p'). set (q'
  := pr1 (pr1 a)). set (r' := pr2 (pr1 a)). assert (n ^> -
  n) as f0. apply pathsinv0. apply rngminusminus. assert (- n
  ^> - (nattohz (hzabsval n))) as f1. apply pathsinv0. apply
  maponpaths. apply (hzabsvalh0). apply hzlthtoleh. assumption.
  destruct (hzlehchoice 0 r' (pr1 (pr2 (pr2 (a))))) as [
  less | equal]. split with (dirprodpair (- q' - 1) (m - r')).
  ). split. change (pr1 (dirprodpair (- q' - 1) (m - r')))
  with (- q' - 1). change (pr2 (dirprodpair (- q' - 1) (m -
  r'))) with (m - r'). change (- q' - 1) with (- q' + (-
  1%hz)). rewrite hzldistr. assert (- nattohz (hzabsval n) ^>
  ((m * (- q')) + m * (- 1%hz)) + (m - r'))) as f2.
  rewrite (pr1 (pr2 (a))). change (pr1 (pr1 a)) with
  q'. change (pr2 (pr1 a)) with r'. rewrite
  hzabsvalgth0. rewrite hzaddinvplus. rewrite (rngmultminus hz
  ). rewrite (hzplusassoc _ (m * (- 1%hz)) _). apply (maponpaths
  (fun x : _ => (- (m * q')) + x)). assert (- m + (m - r') ^>
  (m * (- 1%hz) + (m - r'))) as f3. apply (maponpaths (fun
  x : _ => x + (m - r'))). apply pathsinv0. assert (m * (-
  1%hz) ^> (- (m * 1%hz))) as f30. apply (rngmultminus
  ). assert (- (m * 1) ^> - m) as f31. rewrite hzmultl1. apply
  idpath. rewrite f30. assumption. assert (- r' ^> (- m + (m -
  r'))) as f4. change (- r' ^> (- m + (m - r'))). rewrite
  <- (hzplusassoc). rewrite (hzlminus), (hzplusl0). apply
  idpath. rewrite f4. assumption. assumption. rewrite f0,
  f1. assumption.

```

```

split. change (pr2 (dirprodpair (- q' - 1) (m - r'))) with
  (m - r'). apply hzlthtoleh. rewrite <- (hzrminus r'). apply
  hzlthandplusr. rewrite <- (hzabsvalgeh0 m_nonneg). apply (pr2
  (pr2 (a))). rewrite (hzabsvalgeh0 m_nonneg). assert (
  hzlth (m - r') (m + 0)) as u. apply (hzlthandplusl). apply
  (hzgth0andminus). apply less. rewrite hzplusr0 in
  u. assumption.

```

```

split with (dirprodpair (- q') 0). split. change (pr1 (
  dirprodpair (- q') 0)) with (- q'). change (pr2 (
  dirprodpair (- q') 0)) with 0. assert (- nattohz (hzabsval
  n) ^> (m * - q' + 0)) as f2. rewrite (hzplusr0). rewrite (
  pr1 (pr2 (a))). change (pr1 (pr1 a)) with q'. change (
  pr2 (pr1 a)) with r'. rewrite <- equal. rewrite
  hzplusr0. rewrite hzabsvalgeh0. apply pathsinv0. apply

```

```

rngmultminus. assumption. rewrite f0,
f1. assumption. split. apply (isreflhzleh). rewrite equal. apply
(pr2 (pr2 (pr2 (a))))).

assert empty. apply p. apply pathsinv0.
assumption. contradiction.

set (choice2 := hzlthorgeh m 0). destruct choice2 as [m_neg |
m_nonneg].

(*Case III. Assume n>=0, m<0.*) assert (hzlth 0 (nattohz (hzabsval
m))) as p'. rewrite (hzabsvallth0). rewrite <- (
rngminusminus hz m) in m_neg. set (d:= hzlth0andminus m_neg
). rewrite rngminusminus in d. apply d. assumption. set (a :=
divalgorithmmnonneg (hzabsval n) (hzabsval m) p'). set (q'
:= pr1 (pr1 a)). set (r' := pr2 (pr1 a)). split with (
dirprodpair (- q') r'). split. rewrite <- (hzabsvalgeh0).
rewrite (pr1 (pr2 (a))). change (pr1 (pr1 a)) with
q'. change (pr2 (pr1 a)) with r'. change (pr1 (dirprodpair
(- q') r')) with (- q'). change (pr2 (dirprodpair (- q'
) r')) with r'. rewrite (hzabsvalleh0). apply (maponpaths (
fun x : _ => x + r')). assert (- m * q' ^> - (m * q')) as
f0. apply rnglmultminus. assert (- (m * q') ^> m * (- q'))
as f1. apply pathsinv0. apply rngmultminus. exact (pathscomp0 f0
f1). apply hzlthtoleh. assumption. split. apply (
pr1 (pr2 (pr2 (a)))). apply (pr2 (pr2 (pr2 (a)))).

(*Case IV: n>=0, m>0.*)

assert (hzlth 0 (nattohz (hzabsval m))) as p'. rewrite (
hzabsvalgeh0). destruct (hzneqchoice 0 m) as [l | r]. apply
p. assert empty. apply (isirreflhzgth 0). apply (hzgthgehtrans
0 m 0). assumption. assumption. contradiction.
assumption. assumption. set (a := divalgorithmmnonneg (hzabsval
n) (hzabsval m) p'). set (q' := pr1 (pr1 a)). set (r' :=
pr2 (pr1 a)). split with (dirprodpair q' r'). split.
rewrite <- hzabsvalgeh0. rewrite (pr1 (pr2 (a))). change (
pr1 (pr1 a)) with q'. change (pr2 (pr1 a)) with r'. change
(pr1 (dirprodpair q' r')) with q'. change (pr2 (dirprodpair
q' r')) with r'. rewrite hzabsvalgeh0. apply
idpath. assumption. assumption. split. apply (pr1 (pr2 (pr2 (
a)))). apply (pr2 (pr2 (pr2 (a)))). Defined.

Lemma hzdivhzabsval (a b : hz) (p : hzdiv a b) : hdisj (natleh (
hzabsval a) (hzabsval b)) (hzabsval b ^> 0%nat). Proof. intros
a b p P q. apply (p P). intro t. destruct t as [k f]. unfold
hzdiv0 in f. apply q. apply natdivleh with (hzabsval k). rewrite (
hzabsvalandmult). rewrite f. apply idpath. Defined.

Lemma divalgorithmm (n m : hz) (p : hzneq 0 m) : iscontr (total2 (
fun qr : dirprod hz hz => ((dirprod (n ^> ((m * (pr1 qr)) + (
pr2 qr)) (dirprod (hzleh 0 (pr2 qr)) (hzlth (pr2 qr) (
nattohz (hzabsval m))))) ())). Proof. intros. split with (
divalgorithmmexists n m p). intro t. destruct t as [qr' t'].
destruct qr' as [q' r']. simpl in t'. destruct t' as [f' p2p2t
]. destruct p2p2t as [p1p2p2t p2p2p2t]. destruct divalgorithmmexists
as [qr v]. destruct qr as [q r]. destruct v as [f p2p2dae].
destruct p2p2dae as [p1p2p2dae p2p2p2dae]. simpl in f. simpl in
p1p2p2dae. simpl in p2p2p2dae.

assert (r' ^> r) as h. (*Proof that r' ^> r :*) assert (m * (q
- q') ^> (r' - r)) as h0. change (q - q') with (q + - q').
rewrite (hzldistr). rewrite <- (hzplusr0 (r' - r)).
rewrite <- (hzrminus (m * q')). change (r' - r) with (r'
+ (- r)). rewrite (hzplusassoc r'). change ((m * q') - (m
* q')) with ((m * q') + (- (m * q'))). rewrite <- (

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hzplusassoc (- r)). rewrite (hzpluscomm (- r)). rewrite <- (
hzplusassoc r'). rewrite <- (hzplusassoc r'). rewrite (
hzpluscomm r'). rewrite <- f'. rewrite f. rewrite (hzplusassoc (
m * q)). change (r + - r) with (r - r). rewrite (hzrminus).
rewrite (hzplusr0). rewrite (rngmultminus hz). change (m * q
+ - (m * q')) with ((m * q + - (m * q')) %rng). apply
idpath.

```

```

assert (hdisj (natleh (hzabsval m) (hzabsval (r' - r))) (
hzabsval (r' - r) ^> 0%nat)) as v. apply hzdivhzabsval. intro
P. intro s. apply s. split with (q - q'). unfold
hzdiv0. assumption. assert (isaprop (r' ^> r)) as P. apply (
isasethz). apply (v (hProppair (r' ^> r) P)). intro
s. destruct s as [left | right]. assert (hzlth (nattohz (
hzabsval (r' - r))) (nattohz (hzabsval m))) as u.
destruct (hzgthorleh r' r) as [greater | lesseq]. assert (
hzlth 0 (r' - r)) as e. rewrite <- (hzrminus r). apply
hzlthandplusr. assumption. rewrite (hzabsvalgth0). apply
hzlthminus. apply (p2p2p2t). apply (p2p2p2dae). apply (
p1p2p2dae). apply e. destruct (hzlehchoice r' r lesseq) as [
less | equal]. rewrite hzabsvalandminuspos. rewrite
hzabsvalgth0. apply hzlthminus. apply (p2p2p2dae). apply (
p2p2p2t). apply (p1p2p2t). apply hzlthminusequiv.
assumption. apply (p1p2p2t). apply p1p2p2dae. rewrite
equal. rewrite hzrminus. rewrite hzabsval0. rewrite
nattohzand0. apply hzabsvalneq0. intro Q. apply p. assumption.
assert empty. apply (isirreflhzlth (nattohz (hzabsval m))).
apply (hzlehltztrans _ (nattohz (hzabsval (r' - r))) _).
apply nattohzandleh. assumption. assumption. contradiction.
assert (r' ^> r) as i. assert (r' - r ^> 0) as i0. apply
hzabsvaleq0. assumption. rewrite <- (hzplusl0 r). rewrite <- (
hzplusr0 r'). assert (r' + (r - r) ^> (0 + r)) as i00.
change (r - r) with (r + - r). rewrite (hzpluscomm _ (- r))
). rewrite <- hzplusassoc. apply (maponpaths (fun x : _ => x +
r)). apply i0. exact (transportf (fun x : _ => (r' + x ^> (
0 + r))) (hzrminus r)) i00). apply i.

```

```

assert (q' ^> q) as g. (*Proof that q' ^> q :*) rewrite h in
f'. rewrite f in f'. apply (hzmultlcan q' q m). intro i. apply
p. apply pathsinv0. assumption. apply (hzplusrcan (m * q')) (m *
q) r). apply pathsinv0. apply f'.

```

```

(*Path in direct product :*) assert (dirprodpair q' r' ^> (
dirprodpair q r)) as j. apply
pathsdirdprod. assumption. assumption.

```

```

(*Proof of general path :*) apply pathintotalfiber with (p0 := j
). assert (iscontr (dirprod (n ^> (m * q + r)) (dirprod (
hzleh 0 r) (hzlth r (nattohz (hzabsval m))))) as
contract. change iscontr with (isofhlevel 0). apply
isofhleveldirprod. split with f. intro t. apply isasethz. apply
isofhleveldirprod. split with p1p2p2dae. intro t. apply hzleh.
split with p2p2p2dae. intro t. apply hzlth. apply
proofirrelevancecontr. assumption. Defined.

```

```

Definition hzquotientmod (p : hz) (x : hzneq 0 p) : hz -> hz :=
fun n : hz => (pr1 (pr1 (divalgorithmmexists n p x))).

```

```

Definition hzremaindermod (p : hz) (x : hzneq 0 p) : hz -> hz :=
fun n : hz => (pr2 (pr1 (divalgorithmmexists n p x))).

```

```

Definition hzdividequationmod (p : hz) (x : hzneq 0 p) (n : hz) :
n ^> (p * (hzquotientmod p x n) + (hzremaindermod p x n)) := (
pr1 (pr2 (divalgorithmmexists n p x))).

```

```

Definition hzleh0remaindermod (p : hz) (x : hzneq 0 p) (n : hz)

```

```

: hzleh 0 (hzremaindermod p x n) := (pr1 (pr2 (pr2 (
divalgorithmexists n p x))))).

Definition hzlthremaindermodmod (p : hz) (x : hzneq 0 p) (n : hz)
: hzlth (hzremaindermod p x n) (nattohz (hzabsval p)) := (pr2
(pr2 (pr2 (divalgorithmexists n p x))))).

(* Eval lazy in hzabsval ((hzquotientmod (1 + 1) testlemma2 (1
+ 1 + 1 + 1 + 1 + 1 + 1 + 1))) ). Eval lazy in hzabsval (((
hzremaindermod (1 + 1) testlemma2 (1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 +
1))))). *)

(** * II. QUOTIENTS AND REMAINDERS *)

Definition isaprim (p : hz) : UU := dirprod (hzlth 1 p) (forall
m : hz, (hzdiv m p) -> (hdisj (m ~> 1) (m ~> p))).

Lemma isapropisaprim (p : hz) : isaprop (isaprim p). Proof.
intros. apply isofhleveldirprod. apply (hzlth 1 p). apply
impred. intro m. apply impredfun. apply (hdisj (m ~> 1) (m ~> p))
). Defined.

Lemma isaprimetoneq0 {p : hz} (x : isaprim p) : hzneq 0 p.
Proof. intros. intros f. apply (isirreflhzlth 0). apply (
istranshzlth _ 1 _). apply hzlthnsn. rewrite f. apply (pr1 x).
Defined.

Lemma hzqrtest (m : hz) (x : hzneq 0 m) (a q r : hz) : dirprod (
a ~> ((m * q) + r)) (dirprod (hzleh 0 r) (hzlth r (nattohz
(hzabsval m)))) -> dirprod (q ~> hzquotientmod m x a) (r ~>
hzremaindermod m x a). Proof. intros m x a q r d. set (k := tpair
(P := (fun qr : dirprod hz hz => dirprod (a ~> (m * (pr1 qr) +
pr2 qr)) (dirprod (hzleh 0 (pr2 qr)) (hzlth (pr2 qr) (
nattohz (hzabsval m)))))) (dirprodpair q r) d). assert (k
~> (pr1 (divalgorithm a m x))) as f. apply (pr2 (divalgorithm
a m x)). split. change q with (pr1 (pr1 k)). rewrite f. apply
idpath. change r with (pr2 (pr1 k)). rewrite f. apply idpath.
Defined.

Definition hzqrtestq (m : hz) (x : hzneq 0 m) (a q r : hz) (d :
dirprod (a ~> ((m * q) + r)) (dirprod (hzleh 0 r) (hzlth r (
nattohz (hzabsval m)))))) := pr1 (hzqrtest m x a q r d).

Definition hzqrtestr (m : hz) (x : hzneq 0 m) (a q r : hz) (d :
dirprod (a ~> ((m * q) + r)) (dirprod (hzleh 0 r) (hzlth r (
nattohz (hzabsval m)))))) := pr2 (hzqrtest m x a q r d).

Lemma hzgrand0eq (p : hz) (x : hzneq 0 p) : 0 ~> ((p * 0) + 0).
Proof. intros. rewrite hzmultip0. rewrite hzplusl0. apply idpath.
Defined.

Lemma hzgrand0ineq (p : hz) (x : hzneq 0 p) : dirprod (hzleh 0 0)
(hzlth 0 (nattohz (hzabsval p))). Proof.
intros. split. apply isreflhzleh. apply hzabsvalneq0. assumption.
Defined.

Lemma hzgrand0q (p : hz) (x : hzneq 0 p) : hzquotientmod p x 0 ~>
0. Proof. intros. apply pathsinv0. apply (hzqrtestq p x 0 0 0).
split. apply (hzgrand0eq p x). apply (hzgrand0ineq p x). Defined.

Lemma hzgrand0r (p : hz) (x : hzneq 0 p) : hzremaindermod p x 0 ~>
0. Proof. intros. apply pathsinv0. apply (hzqrtestr p x 0 0 0).
split. apply (hzgrand0eq p x). apply (hzgrand0ineq p x). Defined.

Lemma hzgrand1eq (p : hz) (is : isaprim p) : 1 ~> ((p * 0) + 1).
Proof. intros. rewrite hzmultip0. rewrite hzplusl0. apply idpath.

```

Defined.

```

Lemma hzgrand1eq (p : hz) (is : isaprim p) : dirprod (hzleh 0
1) (hzlth 1 (nattohz (hzabsval p))). Proof.
intros. split. apply hzlthtoleh. apply hzlthnsn. rewrite (
hzabsvalgth0). apply is. apply (istranshzgth _ 1 _). apply
is. apply (hzgthnsn 0). Defined.

```

```

Lemma hzgrand1q (p : hz) (is : isaprim p) : hzquotientmod p (
isaprimetoneq0 is) 1 ~> 0. Proof. intros. apply pathsinv0. apply (
hzqrtestq p (isaprimetoneq0 is) 1 0 1). split. apply (hzgrand1eq
p is). apply (hzgrand1eq p is). Defined.

```

```

Lemma hzgrand1r (p : hz) (is : isaprim p) : hzremaindermod p (
isaprimetoneq0 is) 1 ~> 1. Proof. intros. apply pathsinv0. apply (
hzqrtestr p (isaprimetoneq0 is) 1 0 1). split. apply (hzgrand1eq
p is). apply (hzgrand1eq p is). Defined.

```

```

Lemma hzgrandselfeq (p : hz) (x : hzneq 0 p) : p ~> (p * 1 + 0).
Proof. intros. rewrite hzmultip1. rewrite hzplusr0. apply idpath.
Defined.

```

```

Lemma hzgrandselfeq (p : hz) (x : hzneq 0 p) : dirprod (hzleh 0
0) (hzlth 0 (nattohz (hzabsval p))). Proof. split. apply
isreflhzleh. apply hzabsvalneq0. assumption. Defined.

```

```

Lemma hzgrandselfq (p : hz) (x : hzneq 0 p) : hzquotientmod p x p
~> 1. Proof. intros. apply pathsinv0. apply (hzqrtestq p x p 1 0).
split. apply (hzgrandselfeq p x). apply (hzgrandselfeq p x).
Defined.

```

```

Lemma hzgrandselfr (p : hz) (x : hzneq 0 p) : hzremaindermod p x p
~> 0. Proof. intros. apply pathsinv0. apply (hzqrtestr p x p 1 0).
split. apply (hzgrandselfeq p x). apply (hzgrandselfeq p x).
Defined.

```

```

Lemma hzgrandpluseq (p : hz) (x : hzneq 0 p) (a c : hz) : (a +
c) ~> ((p * (hzquotientmod p x a + hzquotientmod p x c +
hzquotientmod p x (hzremaindermod p x a + hzremaindermod p x c)))
+ hzremaindermod p x ((hzremaindermod p x a) + (hzremaindermod p x
c))). Proof. intros. rewrite 2! (hzldistr). rewrite (
hzplusassoc). rewrite <- (hzdivequationmod p x (hzremaindermod p x
a + hzremaindermod p x c)). rewrite hzplusassoc. rewrite (
hzpluscomm (hzremaindermod p x a)). rewrite <- (hzplusassoc (p *
hzquotientmod p x c)). rewrite <- (hzdivequationmod p x c).
rewrite (hzpluscomm c). rewrite <- hzplusassoc. rewrite <- (
hzdivequationmod p x a). apply idpath. Defined.

```

```

Lemma hzgrandplusineq (p : hz) (x : hzneq 0 p) (a c : hz) :
dirprod (hzleh 0 (hzremaindermod p x (hzremaindermod p x a +
hzremaindermod p x c))) (hzlth (hzremaindermod p x (
hzremaindermod p x a + hzremaindermod p x c)) (nattohz (hzabsval p)
))). Proof. intros. split. apply hzleh0remaindermod. apply
hzlthremaindermodmod. Defined.

```

```

Lemma hzremaindermodandplus (p : hz) (x : hzneq 0 p) (a c : hz) :
hzremaindermod p x (a + c) ~> hzremaindermod p x (hzremaindermod
p x a + hzremaindermod p x c). Proof. intros. apply
pathsinv0. apply (hzqrtest p x (a + c) _ _ (dirprodpair (
hzgrandpluseq p x a c) (hzgrandplusineq p x a c))). Defined.

```

```

Lemma hzquotientmodandplus (p : hz) (x : hzneq 0 p) (a c : hz) :
hzquotientmod p x (a + c) ~> (hzquotientmod p x a + hzquotientmod
p x c + hzquotientmod p x (hzremaindermod p x a + hzremaindermod p x
c)). Proof. intros. apply pathsinv0. apply (hzqrtest p x (a + c)
_ _ (dirprodpair (hzgrandpluseq p x a c) (hzgrandplusineq p x a c

```

)))). Defined.

Lemma hzgrandtimeseq (m : hz) (x : hzneq 0 m) (a b : hz) : (a * b) ^> ((m * (hzquotientmod m x) a * (hzquotientmod m x) b * m + (hzremaindermod m x b) * (hzquotientmod m x a) + (hzremaindermod m x a) * (hzquotientmod m x b) + (hzquotientmod m x (hzremaindermod m x a * hzremaindermod m x b)))) + hzremaindermod m x (hzremaindermod m x a * hzremaindermod m x b)). Proof.
 intros. rewrite 3! (hzldistr). rewrite (hzplusassoc _ _ (hzremaindermod m x (hzremaindermod m x a * hzremaindermod m x b)))).
 rewrite <- hzdivequationmod. rewrite (hzmultassoc _ _ m).
 rewrite <- (hzmultassoc m _ (hzquotientmod m x b * m)). rewrite (hzmultcomm _ m). change ((m * hzquotientmod m x a * (m * hzquotientmod m x b))%hz + m * (hzremaindermod m x b * hzquotientmod m x a)%hz)%rng with ((m * hzquotientmod m x a * (m * hzquotientmod m x b))%hz + m * (hzremaindermod m x b * hzquotientmod m x a)%hz). change (a * b ^> ((m * hzquotientmod m x a * (m * hzquotientmod m x b) + m * (hzremaindermod m x b * hzquotientmod m x a))%hz + m * (hzremaindermod m x a * hzquotientmod m x b)%hz)%rng + hzremaindermod m x a * hzremaindermod m x b) with (a * b ^> ((m * hzquotientmod m x a * (m * hzquotientmod m x b) + m * (hzremaindermod m x b * hzquotientmod m x a)%hz + hzremaindermod m x a * hzremaindermod m x b)). rewrite (hzplusassoc (m * hzquotientmod m x a * (m * hzquotientmod m x b)) _ _).
 rewrite (hzpluscomm (m * (hzremaindermod m x b * hzquotientmod m x a)) (m * (hzremaindermod m x a * hzquotientmod m x b))).
 rewrite <- (hzmultassoc m (hzremaindermod m x a) (hzquotientmod m x b)).
 rewrite (hzmultcomm m (hzremaindermod m x a)).
 rewrite (hzmultassoc (hzremaindermod m x a) m (hzquotientmod m x b)).
 rewrite <- (hzplusassoc (m * hzquotientmod m x a * (m * hzquotientmod m x b)) (hzremaindermod m x a * hzremaindermod m x b) (hzremaindermod m x a * (m * hzquotientmod m x b) + (m * hzquotientmod m x a * hzremaindermod m x b))).
 rewrite <- (hzrdistr). rewrite <- hzdivequationmod. rewrite <- hzldistr. rewrite <- hzdivequationmod. apply idpath. Defined.

Lemma hzgrandtimeseq (m : hz) (x : hzneq 0 m) (a b : hz) :
 dirprod (hzleh 0 (hzremaindermod m x (hzremaindermod m x a * hzremaindermod m x b))) (hzlth (hzremaindermod m x (hzremaindermod m x a * hzremaindermod m x b)) (nattohz (hzabsval m)))). Proof. intros. split. apply hzleh0remaindermod. apply hzlthremaindermodmod. Defined.

Lemma hzquotientmodandtimes (m : hz) (x : hzneq 0 m) (a b : hz) :
 hzquotientmod m x (a * b) ^> ((hzquotientmod m x) a * (hzquotientmod m x) b * m + (hzremaindermod m x b) * (hzquotientmod m x a) + (hzremaindermod m x a) * (hzquotientmod m x b) + (hzquotientmod m x (hzremaindermod m x a * hzremaindermod m x b)))).
 Proof. intros. apply pathsinv0. apply (hzqrtestq m x (a * b)) _ (hzremaindermod m x (hzremaindermod m x a * hzremaindermod m x b))).
 split. apply hzgrandtimeseq. apply hzgrandtimeseq. Defined.

Lemma hzremaindermodandtimes (m : hz) (x : hzneq 0 m) (a b : hz) :
 hzremaindermod m x (a * b) ^> (hzremaindermod m x (hzremaindermod m x a * hzremaindermod m x b)). Proof.
 intros. apply pathsinv0. apply (hzqrtestr m x (a * b)) (hzquotientmod m x a * (hzquotientmod m x) b * m + (hzremaindermod m x b) * (hzquotientmod m x a) + (hzremaindermod m x a) * (hzquotientmod m x b) + (hzquotientmod m x (hzremaindermod m x a * hzremaindermod m x b))) _).
 split. apply hzgrandtimeseq. apply hzgrandtimeseq. Defined.

Lemma hzgrandremaindereq (m : hz) (is : hzneq 0 m) (n : hz) : (

hzremaindermod m is n ^> ((m * (pr1 (dirprodpair 0 (hzremaindermod m is n)))) + (pr2 (dirprodpair (@rngunell1 hz) (hzremaindermod m is n))))). Proof. intros. simpl. rewrite hzmultx0. rewrite hzplusl0. apply idpath. Defined.

Lemma hzgrandremainderineq (m : hz) (is : hzneq 0 m) (n : hz) :
 dirprod (hzleh (@rngunell1 hz) (hzremaindermod m is n)) (hzlth (hzremaindermod m is n) (nattohz (hzabsval m)))). Proof.
 intros. split. apply hzleh0remaindermod. apply hzlthremaindermodmod. Defined.

Lemma hzremaindermoditerated (m : hz) (is : hzneq 0 m) (n : hz) :
 hzremaindermod m is (hzremaindermod m is n) ^> (hzremaindermod m is n). Proof. intros. apply pathsinv0. apply (hzqrtestr m is (hzremaindermod m is n) 0 (hzremaindermod m is n)). split. apply hzgrandremaindereq. apply hzgrandremainderineq. Defined.

Lemma hzgrandremaindereq (m : hz) (is : hzneq 0 m) (n : hz) : 0 ^> hzquotientmod m is (hzremaindermod m is n). Proof.
 intros. apply (hzqrtestq m is (hzremaindermod m is n) 0 (hzremaindermod m is n)). split. apply hzgrandremaindereq. apply hzgrandremainderineq. Defined.

(** * III. THE EUCLIDEAN ALGORITHM *)

Definition iscommonhzdiv (k n m : hz) := dirprod (hzdiv k n) (hzdiv k m).

Lemma isapropiscommonhzdiv (k n m : hz) : isaprop (iscommonhzdiv k n m). Proof. intros. unfold isaprop. apply isofhleveldirprod. apply hzdiv. apply hzdiv. Defined.

Definition hzgcd (n m : hz) : UU := total2 (fun k : hz => dirprod (iscommonhzdiv k n m) (forall l : hz, iscommonhzdiv l n m -> hzleh l k)).

Lemma isaprophzgcd0 (k n m : hz) : isaprop (dirprod (iscommonhzdiv k n m) (forall l : hz, iscommonhzdiv l n m -> hzleh l k)). Proof.
 intros. apply isofhleveldirprod. apply isapropiscommonhzdiv. apply impred. intro t. apply impredfun. apply hzleh. Defined.

Lemma isaprophzgcd (n m : hz) : isaprop (hzgcd n m). Proof.
 intros. intros k l. assert (isofhlevel 2 (hzgcd n m)) as aux.
 apply isofhleveltotal2. apply isasethz. intros x. apply hlevelntosn. apply isofhleveldirprod. apply isapropiscommonhzdiv. apply impred. intro t. apply impredfun. apply (hzleh t x). assert (k ^> l) as f. destruct k as [k pq]. destruct pq as [p q]. destruct l as [l pq]. destruct pq as [p' q']. assert (k ^> l) as f0. apply isantisymmhzleh. apply q'. assumption. apply q. assumption.

apply pathintototalfiber with (p0 := f0). assert (isaprop (dirprod (iscommonhzdiv l n m) (forall x : hz, iscommonhzdiv x n m -> hzleh x l))) as is. apply isofhleveldirprod. apply isapropiscommonhzdiv. apply impred. intro t. apply impredfun. apply (hzleh t l). apply is. split with f. intro q. destruct k as [k pq]. destruct pq as [p q]. destruct l as [l pq]. destruct pq as [p' q']. apply aux. Defined.

(* Euclidean algorithm for calculating the GCD of two numbers (here assumed to be natural numbers (m <= n)) :)

gcd (n, m) := 1. if m = 0, then take n. 2. if m \neq 0, then divide n = q * m + r and take g := gcd (m, r). *)

Lemma hzdivandmultl (a c d : hz) (p : hzdiv d a) : hzdiv d (c * a). Proof. intros. intros P s. apply p. intro k. destruct k as [k f

```
]. apply s. unfold hzdiv0. split with ( c * k ). rewrite ( hzmultcomm
d ). rewrite ( hzmultassoc ). unfold hzdiv0 in f. rewrite (
hzmultcomm k ). rewrite f. apply idpath. Defined.
```

```
Lemma hzdivandmultr ( a c d : hz ) ( p : hzdiv d a ) : hzdiv d ( a * c
). Proof. intros. rewrite hzmultcomm. apply
hzdivandmultl. assumption. Defined.
```

```
Lemma hzdivandminus ( a d : hz ) ( p : hzdiv d a ) : hzdiv d ( - a ).
Proof. intros. intros P s. apply p. intro k. destruct k as [ k f
]. apply s. split with ( - k ). unfold hzdiv0. unfold hzdiv0 in
f. rewrite ( rngmultminus hz ). apply maponpaths. assumption.
Defined.
```

```
Definition natgcd ( m n : nat ) : ( natneq 0%nat n ) -> ( natleh m n )
-> ( hzgcd ( nattohz n ) ( nattohz m ) ). Proof. set ( E := ( fun m
: nat => forall n : nat, ( natneq 0%nat n ) -> ( natleh m n ) ) -> (
hzgcd ( nattohz n ) ( nattohz m ) ) ). assert ( forall x : nat, E x
) as goal. apply stronginduction. (* BASE CASE: *) intros n x0
x1. split with ( nattohz n ). split. unfold iscommonhzdiv.
split. unfold hzdiv. intros P s. apply s. unfold hzdiv0. split with
1. rewrite hzmultl. apply idpath. unfold hzdiv. intros P s. apply
s. unfold hzdiv0. split with 0. rewrite hzmultx0. rewrite
nattohzand0. apply idpath. intros 1 t. destruct t as [ t0 t1
]. destruct ( hzgthorleh 1 0 ) as [ left | right ]. rewrite <-
hzabsvalgth0. apply nattohzandleh. unfold hzdiv in t0. apply t0. intro
t2. destruct t2 as [ k t2 ]. unfold hzdiv0 in t2. assert ( coprod (
natleh ( hzabsval 1 ) n ) ( n ~> 0%nat ) ) as C. apply ( natdivleh (
hzabsval 1 ) ( n ) ( hzabsval k ) ). apply ( isinclnatinj
isinclnattohz ). rewrite nattohzandmult. rewrite 2!
hzabsvalgeh0. assumption. assert ( hzgeh ( 1 * k ) ( 1 * 0 ) ) as i.
rewrite hzmultx0. rewrite t2. change 0 with ( nattohz 0%nat ). apply
nattohzandgeh. apply x1. apply ( hzgehandmultlinv _ _ 1
). assumption. assumption. apply hzgthtogeh. assumption. destruct C
as [ C0 | C1 ]. assumption. assert empty. apply x0. apply
pathsinv0. assumption. contradiction. assumption. apply (
istranshzleh _ 0 _ ). assumption. change 0 with ( nattohz 0%nat
). apply nattohzandleh. assumption. (* INDUCTION CASE: *) intros m p
q. intros n i j.
```

```
assert ( hzlth 0 ( nattohz m ) ) as p'. change 0 with ( nattohz
0%nat ). apply nattohzandlth. apply natneq0togeth0. apply p. set (
a := divalgorithmnonneg n m p' ). destruct a as [ qr a ]. destruct
qr as [ quot rem ]. destruct a as [ f a ]. destruct a as [ a b
]. simpl in b. simpl in f. assert ( natlth ( hzabsval rem ) m )
as p''. rewrite <- ( hzabsvalandnattohz m ). apply
nattohzandlthin. rewrite 2! hzabsvalgeh0. assumption. apply
hzgthtogeh. apply ( hzgthgehtrans _ rem ).
assumption. assumption. assumption. assert ( natleh ( hzabsval rem
) n ) as i''. apply natlthtoleh. apply nattohzandlthin. rewrite
hzabsvalgeh0. apply ( hzlthletrans _ ( nattohz m ) _
). assumption. apply nattohzandleh. assumption. assumption.
assert ( natneq 0%nat m ) as p'''. intro ff. apply p. apply
pathsinv0. assumption. destruct ( q ( hzabsval rem ) p'' m p''' (
natlthtoleh _ _ p'' ) ) as [ rr c ]. destruct c as [ c0 c1 ].
split with rr. split. split. apply ( hzdivlinearcombright (
nattohz n ) ( nattohz m * quot ) ( rem ) rr f ). apply
hzdivandmultr. exact ( pr1 c0 ). rewrite hzabsvalgeh0 in c0. exact
( pr2 c0 ). assumption. exact ( pr1 c0 ).
```

```
intros 1 o. apply c1. split. exact ( pr2 o ). rewrite hzabsvalgeh0.
apply ( hzdivlinearcombleft ( nattohz n ) ( nattohz m * quot ) (
rem ) 1 f ). exact ( pr1 o ). apply hzdivandmultr. exact ( pr2 o
). assumption. assumption. Defined.
```

```
Lemma hzgcdandminusl ( m n : hz ) : hzgcd m n ~> hzgcd ( - m ) n.
```

```
Proof. intros. assert ( hProppair ( hzgcd m n ) ( isaprophzgcd _ _ )
~> ( hProppair ( hzgcd ( - m ) n ) ( isaprophzgcd _ _ ) ) ) as
x. apply uahp. intro i. destruct i as [ a i ]. destruct i as [ i0 i1
]. destruct i0 as [ j0 j1 ]. split with a. split. split. apply
j0. intro k. destruct k as [ k f ]. unfold hzdiv0 in f. intros P s.
apply s. split with ( - k ). unfold hzdiv0. rewrite ( rngmultminus hz
). apply maponpaths. assumption. assumption. intros 1 f. apply i1.
split. apply ( pr1 f ). intro k. destruct k as [ k g ]. unfold hzdiv0
in g. intros P s. apply s. split with ( - k ). unfold hzdiv0.
rewrite ( rngmultminus hz ). rewrite <- ( rngminusminus hz m ). apply
maponpaths. assumption. exact ( pr2 f ). intro i. destruct i as [ a i
]. destruct i as [ i0 i1 ]. destruct i0 as [ j0 j1 ]. split with
a. split. split. apply j0. intro k. destruct k as [ k f ]. unfold
hzdiv0 in f. intros P s. apply s. split with ( - k ). unfold hzdiv0.
rewrite ( rngmultminus hz ). rewrite <- ( rngminusminus hz m ).
apply maponpaths. assumption. assumption. intros 1 f. apply
i1. split. apply ( pr1 f ). intro k. destruct k as [ k g ]. unfold
hzdiv0 in g. intros P s. apply s. split with ( - k ). unfold hzdiv0.
rewrite ( rngmultminus hz ). apply maponpaths. assumption. exact ( pr2
f ). apply ( pathinttotalpr1 x ). Defined.
```

```
Lemma hzgcdsymm ( m n : hz ) : hzgcd m n ~> hzgcd n m. Proof.
intros. assert ( hProppair ( hzgcd m n ) ( isaprophzgcd _ _ ) ~> (
hProppair ( hzgcd n m ) ( isaprophzgcd _ _ ) ) ) as x. apply
uahp. intro i. destruct i as [ a i ]. destruct i as [ i0 i1
]. destruct i0 as [ j0 j1 ]. split with
a. split. split. assumption. assumption. intros 1 o. apply
i1. split. exact ( pr2 o ). exact ( pr1 o ). intro i. destruct i as [
a i ]. destruct i as [ i0 i1 ]. destruct i0 as [ j0 j1 ]. split with
a. split. split. assumption. assumption. intros 1 o. apply
i1. split. exact ( pr2 o ). exact ( pr1 o ). apply ( pathinttotalpr1 x
). Defined.
```

```
Lemma hzgcdandminusr ( m n : hz ) : hzgcd m n ~> hzgcd m ( - n ).
Proof. intros. rewrite 2! ( hzgcdsymm m ). rewrite
hzgcdandminusl. apply idpath. Defined.
```

```
Definition euclidean ( n m : hz ) ( i : hzneq 0 n ) ( p : natleh (
hzabsval m ) ( hzabsval n ) ) : hzgcd n m. Proof. intros. assert (
natneq 0%nat ( hzabsval n ) ) as j. intro x. apply i. assert (
hzabsval n ~> 0%nat ) as f. apply pathsinv0. assumption. rewrite (
hzabsvaleq0 f ). apply idpath. set ( a := natgcd ( hzabsval m ) (
hzabsval n ) j p ). destruct ( hzlthorgeh 0 n ) as [ left_n | right_n
]. destruct ( hzlthorgeh 0 m ) as [ left_m | right_m ]. rewrite 2! (
hzabsvalgth0 ) in a. assumption. assumption. assumption. rewrite
hzabsvalgth0 in a. rewrite hzabsvalleh0 in a. rewrite
hzgcdandminusr. assumption. assumption. assumption. destruct (
hzlthorgeh 0 m ) as [ left_m | right_m ]. rewrite ( hzabsvalgth0
left_m ) in a. rewrite hzabsvalleh0 in a. rewrite
hzgcdandminusl. assumption. assumption. rewrite 2! hzabsvalleh0 in
a. rewrite hzgcdandminusl. rewrite hzgcdandminusr.
assumption. assumption. assumption. Defined.
```

```
Theorem euclideanalgorithm ( n m : hz ) ( i : hzneq 0 n ) : iscontr (
hzgcd n m ). Proof. intros. destruct ( natgthorleh ( hzabsval m ) (
hzabsval n ) ) as [ left | right ]. assert ( hzneq 0 m ) as i'. intro
f. apply ( negnatlthn0 ( hzabsval n ) ). rewrite <- f in
left. rewrite hzabsval0 in left. assumption. set ( a := ( euclidean m
n i' ( natlthtoleh _ _ left ) ) ). rewrite hzgcdsymm in a. split with
a. intro. apply isaprophzgcd. split with ( euclidean n m i right
). intro. apply isaprophzgcd. Defined.
```

```
Definition gcd ( n m : hz ) ( i : hzneq 0 n ) : hz := pr1 ( pr1 (
euclideanalgorithm n m i ) ).
```

```
Definition gcdiscommondiv ( n m : hz ) ( i : hzneq 0 n ) := pr1 ( pr2
```

(pr1 (euclideanalgorithm n m i))).

Definition gcdisgreatest (n m : \mathbb{Z}) (i : $\text{hzneq } 0 \text{ n}$) := pr2 (pr2 (pr1 (euclideanalgorithm n m i))).

Lemma hzdivand0 (n : \mathbb{Z}) : hzdiv n 0. Proof. intros. intros P s. apply s. split with 0. unfold hzdiv0. apply hzmultx0. Defined.

Lemma nozerodiv (n : \mathbb{Z}) (i : $\text{hzneq } 0 \text{ n}$) : neg (hzdiv 0 n). Proof. intros. intro p. apply i. apply (p (hProppair (0 \rightarrow n) (isasetz 0 n))). intro t. destruct t as [k f]. unfold hzdiv0 in f. rewrite (hzmult0x) in f. assumption. Defined.

(** * IV. Bezout's lemma and the commutative ring $\mathbb{Z}/p\mathbb{Z}$ *)

Lemma commonhzdivsignswap (k n m : \mathbb{Z}) (p : iscommonhzdiv k n m) : iscommonhzdiv (-k) n m. Proof. intros. destruct p as [p0 p1]. split. apply p0. intro t. intros P s. apply s. destruct t as [l f]. unfold hzdiv0 in f. split with (-1). unfold hzdiv0. change (k * l) with (k * l)%rng in f. rewrite <- rngmultminusminus in f. assumption. apply p1. intro t. destruct t as [l f]. unfold hzdiv0 in f. intros P s. apply s. split with (-1). unfold hzdiv0. change (k * l) with (k * l)%rng in f. rewrite <- rngmultminusminus in f. assumption. Defined.

Lemma gcdneq0 (n m : \mathbb{Z}) (i : $\text{hzneq } 0 \text{ n}$) : $\text{hzneq } 0$ (gcd n m i). Proof. intros. intro f. apply (nozerodiv n). assumption. rewrite f. exact (pr1 (gcdiscommondiv n m i)). Defined.

Lemma gcdpositive (n m : \mathbb{Z}) (i : $\text{hzneq } 0 \text{ n}$) : hzlth 0 (gcd n m i). Proof. intros. destruct (hzneqchoice 0 (gcd n m i) (gcdneq0 n m i)) as [left | right]. assert empty. assert (hzleh (- (gcd n m i)) (gcd n m i)) as i0. apply (gcdisgreatest n m i). apply commonhzdivsignswap. exact (gcdiscommondiv n m i). apply (isirreflhzlth 0). apply (istranshzlth _ (- (gcd n m i)) _). apply hzlth0andminus. assumption. apply (hzlehlthtrans _ (gcd n m i) _). assumption. assumption. assumption. Defined.

Lemma gcdanddiv (n m : \mathbb{Z}) (i : $\text{hzneq } 0 \text{ n}$) (p : hzdiv n m) : coprod (gcd n m i \rightarrow n) (gcd n m i \rightarrow -n). Proof. intros. destruct (hzneqchoice 0 n i) as [left | right]. apply ii2. apply isantisymmhzleh. apply (hzdivhzabsval (gcd n m i) n (pr1 (gcdiscommondiv n m i))). intro c'. destruct c' as [c0 | c1]. rewrite <- (hzabsvalgeh0). rewrite <- (hzabsvallth0). apply nattohzandleh. assumption. assumption. apply hzgthtogeh. apply (gcdpositive n m i). assert empty. assert (n \rightarrow 0) as f. rewrite hzabsvaleq0. apply idpath. assumption. apply i. apply pathsinv0. assumption. contradiction. apply (pr2 (pr2 (pr1 (euclideanalgorithm n m i)))). apply commonhzdivsignswap. split. apply hzdivisrefl. assumption. apply iiii. apply isantisymmhzleh. apply (hzdivhzabsval (gcd n m i) n (pr1 (gcdiscommondiv n m i))). intro c'. destruct c' as [c0 | c1]. rewrite <- hzabsvalgth0. assert (n \rightarrow nattohz (hzabsval n)) as f. apply pathsinv0. apply hzabsvalgth0. assumption. assert (hzleh (nattohz (hzabsval (gcd n m i))) (nattohz (hzabsval n))) as j. apply nattohzandleh. assumption. exact (transportf (fun x : $_ \rightarrow$ hzleh (nattohz (hzabsval (gcd n m i))) x) (pathsinv0 f) j). apply gcdpositive. assert empty. apply i. apply pathsinv0. rewrite hzabsvaleq0. apply idpath. assumption. contradiction. apply (gcdisgreatest n m i). split. apply hzdivisrefl. assumption. Defined.

Lemma gcdand0 (n : \mathbb{Z}) (i : $\text{hzneq } 0 \text{ n}$) : coprod (gcd n 0 i \rightarrow n) (gcd n 0 i \rightarrow -n). Proof. intros. apply gcdanddiv. apply hzdivand0. Defined.

Lemma natbezoutstrong (m n : nat) (i : $\text{hzneq } 0$ (nattohz n)) :

total2 (fun ab : dirprod hz hz => (gcd (nattohz n) (nattohz m) i \rightarrow ((pr1 ab) * (nattohz n) + (pr2 ab) * (nattohz m))))). Proof. set (E := (fun m : nat => forall n : nat, forall i : $\text{hzneq } 0$ (nattohz n), total2 (fun ab : dirprod hz hz => gcd (nattohz n) (nattohz m) i \rightarrow ((pr1 ab) * (nattohz n) + (pr2 ab) * (nattohz m))))). assert (forall x : nat, E x) as goal. apply stronginduction. (* Base Case: *) unfold E. intros. split with (dirprodpair 1 0). simpl. rewrite nattohzand0. destruct (gcdand0 (nattohz n) i) as [left | right]. rewrite hzmultl1. rewrite hzplusr0. assumption. assert empty. apply (isirreflhzlth (gcd (nattohz n) 0 i)). apply (istranshzlth _ 0 _). rewrite right. apply hzgth0andminus. change 0 with (nattohz 0%nat). apply nattohzandgth. apply natneq0togth0. intro f. apply i. rewrite f. apply idpath. apply gcdpositive. contradiction. (* Induction Case: *) intros m x y. intros n i. assert (hzneq 0 (nattohz m)) as p. intro f. apply x. apply pathsinv0. rewrite <- hzabsvalandnattohz. change 0%nat with (hzabsval (nattohz 0%nat)). apply maponpaths. assumption. set (r := hzremaindermod (nattohz m) p (nattohz n)). set (q := hzquotientmod (nattohz m) p (nattohz n)). assert (natlth (hzabsval r) m) as p'. rewrite <- (hzabsvalandnattohz m). apply hzabsvalandlth. exact (hzleh0remaindermod (nattohz m) p (nattohz n)). unfold r. unfold hzremaindermod. rewrite <- (hzabsvalgeh0 (pr1 (pr2 (pr2 (divalgorithmeexists (nattohz n) (nattohz m) p))))). apply nattohzandlth. assert (natlth (hzabsval (pr2 (pr1 (divalgorithmeexists (nattohz n) (nattohz m) p)))) (hzabsval (nattohz m))) as ii. apply hzabsvalandlth. exact (hzleh0remaindermod (nattohz m) p (nattohz n)). assert (nattohz (hzabsval (nattohz m)) \rightarrow (nattohz m)) as f. apply maponpaths. apply hzabsvalandnattohz. exact (transportf (fun x : $_ \rightarrow$ hzlth (pr2 (pr1 (divalgorithmeexists (nattohz n) (nattohz m) p))) x) f (pr2 (pr2 (divalgorithmeexists (nattohz n) (nattohz m) p))))). exact (transportf (fun x : $_ \rightarrow$ natlth (hzabsval (pr2 (pr1 (divalgorithmeexists (nattohz n) (nattohz m) p)))) x) (hzabsvalandnattohz m) ii). set (c := y (hzabsval r) p' m p). destruct c as [ab f]. destruct ab as [a b]. simpl in f. (* split with (dirprodpair ((nattohz n) - q * (nattohz m)) (a - b * q)).*) split with (dirprodpair b (a - b * q)). assert (gcd (nattohz m) (nattohz (hzabsval r)) p \rightarrow (gcd (nattohz n) (nattohz m) i)) as g. apply isantisymmhzleh. apply (gcdisgreatest (nattohz n) (nattohz m) i). split. apply (hzdivlinearcombright (nattohz n) ((nattohz m) * (hzquotientmod (nattohz m) p (nattohz n)) r). exact (hzdivequationmod (nattohz m) p (nattohz n)). apply hzdivandmultr. apply gcdiscommondiv. unfold r. rewrite (hzabsvalgeh0 (hzleh0remaindermod (nattohz m) p (nattohz n))). apply (pr2 (gcdiscommondiv (nattohz m) (hzremaindermod (nattohz m) p (nattohz n) p))). apply gcdiscommondiv. apply gcdisgreatest. split. apply (pr2 (gcdiscommondiv _ _)). apply (hzdivlinearcombleft (nattohz n) ((nattohz m) * (hzquotientmod (nattohz m) p (nattohz n))) (nattohz (hzabsval r)))). unfold r. rewrite (hzabsvalgeh0 (hzleh0remaindermod (nattohz m) p (nattohz n))). exact (hzdivequationmod (nattohz m) p (nattohz n)). apply gcdiscommondiv. apply (hzdivandmultr). apply (pr2 (gcdiscommondiv _ _)). rewrite <- g. rewrite f. simpl. assert (nattohz (hzabsval r) \rightarrow ((nattohz n) - (q * nattohz m))) as h. rewrite (hzdivequationmod (nattohz m) p (nattohz n)). change (hzquotientmod (nattohz m) p (nattohz n)) with q. change (hzremaindermod (nattohz m) p (nattohz n)) with r. rewrite hzpluscomm. change (r + nattohz m * q - q * nattohz m) with ((r + nattohz m * q) + (- (q * nattohz m))). rewrite hzmultcomm. rewrite hzplusassoc. change (q * nattohz m + - (q * nattohz m)) with ((q * nattohz m - (q * nattohz m))). rewrite hzrminus. rewrite hzplusr0. apply hzabsvalgeh0. apply (hzleh0remaindermod (nattohz m) p (nattohz n)). rewrite h. change ((nattohz n - q * nattohz m) with ((nattohz n + (- (q * nattohz

```

m) )) at 1. rewrite (rngldistr hz). rewrite <- (hzplusassoc).
rewrite (hzpluscomm (a * nattohz m)). rewrite
rngmultminus. rewrite <- hzmultipassoc. rewrite <-
rnglmultminus. rewrite hzplusassoc. rewrite <- (rngrdistr hz).
change (b * nattohz n + (a - b * q) * nattohz m) with ((b * nattohz
n)%rng + ((a + - (b * q)%hz) * nattohz m)%rng). apply idpath. apply
goal. Defined.

```

```

Lemma divandhzabsval (n : hz) : hzdiv n (nattohz (hzabsval n)) .
Proof. intros. destruct (hzlthorgeh 0 n) as [left | right].
intros P s. apply s. split with 1. unfold hzdiv0. rewrite hzmultl1.
rewrite hzabsvalgth0. apply idpath. assumption. intros P s. apply
s. split with (- 1%hz). unfold hzdiv0. rewrite (rngmultminus hz).
rewrite hzmultl1. rewrite hzabsvalleh0. apply idpath. assumption.
Defined.

```

```

Lemma bezoutstrong (m n : hz) (i : hzneq 0 n) : total2 (fun ab :
dirprod hz hz => (gcd n m i ^> ((pr1 ab) * n + (pr2 ab) * m))
). Proof. intros. assert (hzneq 0 (nattohz (hzabsval n))) as
i'. intro f. apply i. destruct (hzneqchoice 0 n i) as [left |
right]. rewrite hzabsvallth0 in f. rewrite <- (rngminusminus hz
). change 0 with (- - 0). apply
maponpaths. assumption. assumption. rewrite hzabsvalgth0 in
f. assumption. assumption. set (c := (natbezoutstrong (hzabsval m)
(hzabsval n) i')). destruct c as [ab f]. destruct ab as [a b
]. simpl in f. assert (gcd n m i ^> gcd (nattohz (hzabsval n)) (
nattohz (hzabsval m)) i') as g. destruct (hzneqchoice 0 n i) as
[ left_n | right_n ]. apply isantisymhzhleh. apply
gcdisgreatest. split. rewrite hzabsvallth0. apply
hzdivandminus. apply gcdiscomondiv. assumption. destruct (
hzlthorgeh 0 m) as [left_m | right_m]. rewrite hzabsvalgth0.
apply (pr2 (gcdiscomondiv _ _ _)). assumption. rewrite
hzabsvalleh0. apply hzdivandminus. apply (pr2 (gcdiscomondiv _ _
)). assumption. apply gcdisgreatest. split. apply (hzdivistrans _
(nattohz (hzabsval n)) _). apply gcdiscomondiv. rewrite
hzabsvallth0. rewrite <- (rngminusminus hz n). apply
hzdivandminus. rewrite (rngminusminus hz n). apply hzdivisrefl.
assumption. apply (hzdivistrans _ (nattohz (hzabsval m)) _).
apply (pr2 (gcdiscomondiv _ _ _)). destruct (hzlthorgeh 0 m)
as [left_m | right_m]. rewrite hzabsvalgth0. apply
hzdivisrefl. assumption. rewrite hzabsvalleh0. rewrite <- (
rngminusminus hz m). apply hzdivandminus. rewrite (rngminusminus hz
m). apply hzdivisrefl. assumption. apply isantisymhzhleh. apply
gcdisgreatest. split. rewrite hzabsvalgth0. apply
gcdiscomondiv. assumption. apply (hzdivistrans _ (nattohz (
hzabsval m)) _). destruct (hzlthorgeh 0 m) as [left_m | right_m
]. rewrite hzabsvalgth0. apply (pr2 (gcdiscomondiv _ _ _)).
assumption. rewrite hzabsvalleh0. apply hzdivandminus. apply (pr2 (
gcdiscomondiv _ _ _)). assumption. apply hzdivisrefl. apply
gcdisgreatest. split. apply (hzdivistrans _ (nattohz (hzabsval n)
)). apply gcdiscomondiv. rewrite hzabsvalgth0. apply
hzdivisrefl. assumption. apply (hzdivistrans _ (nattohz (hzabsval
m)) _). apply (pr2 (gcdiscomondiv _ _ _)). destruct (
hzlthorgeh 0 m) as [left_m | right_m]. rewrite hzabsvalgth0. apply
hzdivisrefl. assumption. rewrite hzabsvalleh0. rewrite <- (
rngminusminus hz m). apply hzdivandminus. rewrite (rngminusminus hz
m). apply hzdivisrefl. assumption. destruct (hzneqchoice 0 n i) as
[ left_n | right_n ]. destruct (hzlthorgeh 0 m) as [left_m |
right_m ].

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split with (dirprodpair (- a) b). simpl. assert (- a * n + b *
m ^> (a * (nattohz (hzabsval n)) + b * (nattohz (hzabsval m)
))) as l. rewrite hzabsvallth0. rewrite hzabsvalgth0. rewrite (
rnglmultminus hz). rewrite <- (rngmultminus hz). apply
idpath. assumption. assumption. rewrite l. rewrite g. exact
f. split with (dirprodpair (- a) (- b)). simpl. rewrite 2! (

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rnglmultminus hz). rewrite <- 2! (rngmultminus hz). rewrite <-
(hzabsvallth0). rewrite <- (hzabsvalleh0). rewrite g. exact
f. assumption. assumption. destruct (hzlthorgeh 0 m) as [left_m
| right_m]. split with (dirprodpair a b). simpl. rewrite
g. rewrite f. rewrite 2! hzabsvalgth0. apply
idpath. assumption. assumption. split with (dirprodpair a (- b)
). rewrite g. rewrite f. simpl. rewrite hzabsvalgth0. rewrite
hzabsvalleh0. rewrite (rngmultminus hz). rewrite <- (
rnglmultminus hz). apply idpath. assumption. assumption. Defined.

```

(** * V. Z/nZ *)

```

Lemma hzmodisaprop (p : hz) (x : hzneq 0 p) (nm : hz) : isaprop
(hzremaindermod p x n ^> (hzremaindermod p x m)). Proof.
intros. apply isasetzh. Defined.

```

```

Definition hzmod (p : hz) (x : hzneq 0 p) : hz -> hz -> hProp.
Proof. intros p x n m. exact (hProppair (hzremaindermod p x n ^> (
hzremaindermod p x m)) (hzmodisaprop p x n m)). Defined.

```

```

Lemma hzmodisrefl (p : hz) (x : hzneq 0 p) : isrefl (hzmod p x).
Proof. intros. unfold isrefl. intro n. unfold hzmod. assert (
hzremaindermod p x n ^> (hzremaindermod p x n)) as a. auto. apply
a. Defined.

```

```

Lemma hzmodissymm (p : hz) (x : hzneq 0 p) : issymm (hzmod p x).
Proof. intros. unfold issymm. intros n m. unfold hzmod. intro v.
assert (hzremaindermod p x m ^> hzremaindermod p x n) as a. exact (
pathsinv0 (v)). apply a. Defined.

```

```

Lemma hzmodistrans (p : hz) (x : hzneq 0 p) : istrans (hzmod p x)
). Proof. intros. unfold istrans. intros n m k. intros u v. unfold
hzmod. unfold hzmod in u. unfold hzmod in v. assert (hzremaindermod
p x n ^> hzremaindermod p x k) as a. exact (pathscomp0 u v). apply
a. Defined.

```

```

Lemma hzmodiseqrel (p : hz) (x : hzneq 0 p) : iseqrel (hzmod p x)
). Proof. intros. apply iseqrelconstr. exact (hzmodistrans p x
). exact (hzmodisrefl p x). exact (hzmodissymm p x). Defined.

```

```

Lemma hzmodcompatmultl (p : hz) (x : hzneq 0 p) : forall a b c :
hz, hzmod p x a b -> hzmod p x (c * a) (c * b). Proof. intros p
x a b c v. unfold hzmod. change (hzremaindermod p x (c * a) ^>
hzremaindermod p x (c * b)). rewrite hzremaindermodandtimes. rewrite
v. rewrite <- hzremaindermodandtimes. apply idpath. Defined.

```

```

Lemma hzmodcompatmultr (p : hz) (x : hzneq 0 p) : forall a b c :
hz, hzmod p x a b -> hzmod p x (a * c) (b * c). Proof. intros p
x a b c v. rewrite hzmultipassoc. rewrite (hzmultipassoc b). apply
hzmodcompatmultl. assumption. Defined.

```

```

Lemma hzmodcompatplusl (p : hz) (x : hzneq 0 p) : forall a b c :
hz, hzmod p x a b -> hzmod p x (c + a) (c + b). Proof. intros p
x a b c v. unfold hzmod. change (hzremaindermod p x (c + a) ^>
hzremaindermod p x (c + b)). rewrite
hzremaindermodandplus. rewrite v. rewrite <-
hzremaindermodandplus. apply idpath. Defined.

```

```

Lemma hzmodcompatplusr (p : hz) (x : hzneq 0 p) : forall a b c :
hz, hzmod p x a b -> hzmod p x (a + c) (b + c). Proof. intros p
x a b c v. rewrite hzpluscomm. rewrite (hzpluscomm b). apply
hzmodcompatplusl. assumption. Defined.

```

```

Lemma hzmodisrangeqrel (p : hz) (x : hzneq 0 p) : rangeqrel (X :=
hz). Proof. intros. split with (tpair (hzmod p x) (hzmodiseqrel
p x)). split. split. apply hzmodcompatplusl. apply

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```

hzmmodcompatplusr. split. apply hzmmodcompatmultl. apply
hzmmodcompatmultr. Defined.

```

```

Definition hzmmodp (p : hz) (x : hzneq 0 p) := commrngquot (
hzmmodisrngeqrel p x).

```

```

Lemma isdeceqhzmodp (p : hz) (x : hzneq 0 p) : isdeceq (hzmmodp p
x). Proof. intros. apply (isdeceqsetquot (hzmmodisrngeqrel p x)).
intros a b. unfold isdecprop. destruct (isdeceqhz (
hzremaindermod p x a) (hzremaindermod p x b)) as [l | r].
unfold hzmmodisrngeqrel. simpl. split with (ii1 l). intros t.
destruct t as [f | g]. apply maponpaths. apply isasethz. assert
empty. apply g. assumption. contradiction. split with (ii2 r
). intros t. destruct t as [f | g]. assert empty. apply r.
assumption. contradiction. apply maponpaths. apply isapropneg.
Defined.

```

```

Definition acommrng_hzmmod (p : hz) (x : hzneq 0 p) : acommrng.
Proof. intros. split with (hzmmodp p x). split with (tpair _ (
deceqtoneqapart (isdeceqhzmodp p x))). split. split. intros a b c
q. simpl. simpl in q. intro f. apply q. rewrite f. apply idpath.
intros a b c q. simpl in q. simpl. intro f. apply q. rewrite f. apply
idpath. split. intros a b c q. simpl in q. simpl. intros f. apply
q. rewrite f. apply idpath. intros a b c q. simpl. simpl in q. intro
f. apply q. rewrite f. apply idpath. Defined.

```

```

Lemma hzremaindermodanddiv (p : hz) (x : hzneq 0 p) (a : hz) (y
: hzdiv p a) : hzremaindermod p x a ~> 0. Proof. intros. assert (
isaprop (hzremaindermod p x a ~> 0)) as v. apply isasethz. apply (
y (hProppair _ v)). intro t. destruct t as [k f]. unfold hzdiv0
in f. assert (a ~> (p * k + 0)) as f'. rewrite f. rewrite
hzplus0. apply idpath. set (e := tpair (P := (fun qr : dirprod hz
hz => dirprod (a ~> (p * pr1 qr + pr2 qr)) (dirprod (hzleq 0 (pr2 qr))
(hzleth (pr2 qr) (nattohz (hzabsval p)))))) (dirprodpair k 0) (
dirprodpair f' (dirprodpair (isreflhzleq 0) (hzabsvalneq0 p x))
)). assert (e ~> (pr1 (divalgorithma p x))) as s. apply (pr2 (
divalgorithma p x)). set (w := pathintotalpr1 (pathsinv0 s)
). unfold e in w. unfold hzremaindermod. apply (maponpaths (fun z :
dirprod hz hz => pr2 z) w). Defined.

```

```

Lemma gcdandprime (p : hz) (x : hzneq 0 p) (y : isaprimetoneq0 p) (a
: hz) (q : neg (hzmmod p x a 0)) : gcd p a x ~> 1. Proof.
intros. assert (isaprop (gcd p a x ~> 1)) as is. apply (isasethz
). apply (pr2 y (gcd p a x) (pr1 (gcdiscommoddiv p a x)) (
hProppair _ is)). intro t. destruct t as [t0 | t1]. apply
t0. assert empty. apply q. simpl. assert (hzremaindermod p x a ~> 0)
as f. assert (hzdiv p a) as u. rewrite <- t1. apply (pr2 (
gcdiscommoddiv _ _ _)). rewrite hzremaindermodanddiv. apply
idpath. assumption. rewrite f. rewrite hzgrand0r. apply
idpath. contradiction. Defined.

```

```

Lemma hzremaindermodandmultl (p : hz) (x : hzneq 0 p) (a b : hz)
: hzremaindermod p x (p * a + b) ~> hzremaindermod p x b. Proof.
intros. assert (p * a + b ~> (p * (a + hzquotientmod p x b) +
hzremaindermod p x b)) as f. rewrite hzldistr. rewrite
hzplusassoc. rewrite <- (hzdivequationmod p x b). apply
idpath. rewrite hzremaindermodandplus. rewrite
hzremaindermodandtimes. rewrite hzgrandselfr. rewrite
hzmult0x. rewrite hzgrand0r. rewrite hzplus10. rewrite
hzremaindermoditerated. apply idpath. Defined.

```

```

Lemma hzmmodprimeinv (p : hz) (x : hzneq 0 p) (y : isaprimetoneq0 p) (
a : hz) (q : neg (hzmmod p x a 0)) : total2 (fun v : hz =>
dirprod (hzmmod p x (a * v) 1) (hzmmod p x (v * a) 1)). Proof.
intros. split with (pr2 (pr1 (bezoutstrong a p x))). assert (1
~> (pr1 (pr1 (bezoutstrong a p x)) * p + pr2 (pr1 (bezoutstrong a p
x)) * a)) as f'. assert (1 ~> gcd p a x) as f''. apply
pathsinv0. apply (gcdandprime). assumption. assumption. rewrite
f''. apply (bezoutstrong a p x). split. rewrite f'. simpl. rewrite
(hzmultcomm (pr1 (pr1 (bezoutstrong a p x)))) _). rewrite (
hzremaindermodandmultl). rewrite hzmultcomm. apply idpath. rewrite
f'. simpl. rewrite hzremaindermodandplus. rewrite (
hzremaindermodandtimes p x _ p). rewrite hzgrandselfr. rewrite
hzmultx0. rewrite hzgrand0r. rewrite hzplus10. rewrite
hzremaindermoditerated. apply idpath. Defined.

```

```

Lemma quotientrngsumdecom (X : commrng) (R : rngeqrel (X := X))
(a b : X) : @op2 (commrngquot R) (setquotpr R a) (setquotpr R b)
~> (setquotpr R (a * b) % rng). Proof. intros. auto. Defined.

```

```

Definition ahzmmod (p : hz) (y : isaprimetoneq0 y) : afld. Proof.
intros. split with (acommrng_hzmmod p (isaprimetoneq0 y)).
split. simpl. intro f. apply (isirreflhzleth 0). assert (hzleth 0
1) as i. apply hzlethnsn. change (1 % rng) with (setquotpr (
hzmmodisrngeqrel p (isaprimetoneq0 y)) 1 % hz) in f. change (0 % rng)
with (setquotpr (hzmmodisrngeqrel p (isaprimetoneq0 y)) 0 % hz
). assert (hzmmodisrngeqrel p (isaprimetoneq0 y)) 1 % hz 0 % hz as
o. apply (setquotprpathsandR (hzmmodisrngeqrel p (isaprimetoneq0 y)
) 1 % hz 0 % hz). assumption. unfold hzmmodisrngeqrel in o. simpl in o.
assert (hzremaindermod p (isaprimetoneq0 y) 0 ~> 0) as o'. rewrite
hzgrand0r. apply idpath. rewrite o' in o. assert (hzremaindermod p (
isaprimetoneq0 y) 1 ~> 1) as o''. assert (hzleth 1 p) as v. apply
y. rewrite hzgrand1r. apply idpath. rewrite o'' in o. assert (hzleth
0 1) as o'''. apply hzlethnsn. rewrite o in o'''. assumption. assert
(forall x0 : acommrng_hzmmod p (isaprimetoneq0 y), isaprop ((x0 #
0) % rng ~> multinvpair (acommrng_hzmmod p (isaprimetoneq0 y)) x0))
as int. intro a. apply impred. intro q. apply isapropmultinvpair.
apply (setquotunivprop _ (fun x0 => hProppair _ (int x0))).
intro a. simpl. intro q. assert (neg (hzmmod p (isaprimetoneq0 y)
a 0)) as q'. intro g. unfold hzmmod in g. simpl in g. apply q.
change (0 % rng) with (setquotpr (hzmmodisrngeqrel p (isaprimetoneq0
y)) 0 % hz). apply (iscompsetquotpr (hzmmodisrngeqrel p (
isaprimetoneq0 y))). apply g. split with (setquotpr (
hzmmodisrngeqrel p (isaprimetoneq0 y)) (pr1 (hzmmodprimeinv p (
isaprimetoneq0 y) y a q'))). split. simpl. rewrite (
quotientrngsumdecom hz (hzmmodisrngeqrel p (isaprimetoneq0 y))).
change 1 % multmonoid with (setquotpr (hzmmodisrngeqrel p (
isaprimetoneq0 y)) 1 % hz). apply (iscompsetquotpr (
hzmmodisrngeqrel p (isaprimetoneq0 y))). simpl. change (pr2 (pr1
(bezoutstrong a p (isaprimetoneq0 y))) * a) % rng with (pr2 (pr1
(bezoutstrong a p (isaprimetoneq0 y))) * a) % hz. exact ((pr2 (pr2
(hzmmodprimeinv p (isaprimetoneq0 y) y a q')))). simpl. rewrite (
quotientrngsumdecom hz (hzmmodisrngeqrel p (isaprimetoneq0 y))).
change 1 % multmonoid with (setquotpr (hzmmodisrngeqrel p (
isaprimetoneq0 y)) 1 % hz). apply (iscompsetquotpr (
hzmmodisrngeqrel p (isaprimetoneq0 y))). change (a * pr2 (pr1
(bezoutstrong a p (isaprimetoneq0 y)))) % rng with (a * pr2 (pr1
(bezoutstrong a p (isaprimetoneq0 y)))) % hz. exact ((pr1 (pr2 (
hzmmodprimeinv p (isaprimetoneq0 y) y a q')))). Defined.

```

```

Close Scope hz_scope.
(** END OF FILE *)

```

7.7 The file padics.v

```

(** *p adic numbers *)

(** By Alvaro Pelayo, Vladimir Voevodsky and Michael A. Warren *)

(** 2012 *)

(** Settings *)

Add Rec LoadPath "../Generalities". Add Rec LoadPath "../hlevel1".
Add Rec LoadPath "../hlevel2". Add Rec LoadPath
"../Proof_of_Extensionality". Add Rec LoadPath "../Algebra".

Unset Automatic Introduction. (** This line has to be removed for the
file to compile with Coq8.2 *)

(** Imports *)

Require Export lemmas.

Require Export fps.

Require Export frac.

Require Export z_mod_p.

(** * I. Several basic lemmas *)

Open Scope hz_scope.

Lemma hzgrandnatsummation0r (m : hz) (x : hzneq 0 m) (a : nat ->
hz) (upper : nat) : hzremaindermod m x (natsummation0 upper a) ~>
hzremaindermod m x (natsummation0 upper (fun n : nat =>
hzremaindermod m x (a n))). Proof. intros. induction
upper. simpl. rewrite hzremaindermoditerated. apply idpath. change (
hzremaindermod m x (natsummation0 upper a + a (S upper))) ~>
hzremaindermod m x (natsummation0 upper (fun n : nat =>
hzremaindermod m x (a n) + hzremaindermod m x (a (S upper))))
). rewrite hzremaindermodandplus. rewrite IHupper. rewrite <- (
hzremaindermoditerated m x (a (S upper))). rewrite <-
hzremaindermodandplus. rewrite hzremaindermoditerated. apply idpath.
Defined.

Lemma hzgrandnatsummation0q (m : hz) (x : hzneq 0 m) (a : nat ->
hz) (upper : nat) : hzquotientmod m x (natsummation0 upper a) ~>
(natsummation0 upper (fun n : nat => hzquotientmod m x (a n))) +
hzquotientmod m x (natsummation0 upper (fun n : nat =>
hzremaindermod m x (a n))). Proof. intros. induction
upper. simpl. rewrite <- hzgrandremainderq. rewrite hzplusr0. apply
idpath. change (natsummation0 (S upper) a) with (natsummation0
upper a + a (S upper)). rewrite hzquotientmodandplus. rewrite
IHupper. rewrite (hzplusassoc (natsummation0 upper (fun n : nat =>
hzquotientmod m x (a n))) _ (hzquotientmod m x (a (S upper))
)). rewrite (hzpluscomm (hzquotientmod m x (natsummation0 upper (
fun n : nat => hzremaindermod m x (a n))) (hzquotientmod m x (a
(S upper))))). rewrite <- (hzplusassoc (natsummation0 upper (
fun n : nat => hzquotientmod m x (a n))) (hzquotientmod m x (a
(S upper))))). change (natsummation0 upper (fun n : nat =>
hzquotientmod m x (a n) + hzquotientmod m x (a (S upper))))
with (natsummation0 (S upper) (fun n : nat => hzquotientmod m x (
a n))).

rewrite hzgrandnatsummation0r. rewrite hzquotientmodandplus.
rewrite <- hzgrandremainderq. rewrite hzplusr0. rewrite
hzremaindermoditerated. rewrite (hzplusassoc (natsummation0 (S upper)
) (fun n : nat => hzquotientmod m x (a n))) (hzquotientmod m x (
natsummation0 upper (fun n : nat => hzremaindermod m x (a n))))

```

```

_ ). rewrite <- (hzplusassoc (hzquotientmod m x (natsummation0
upper (fun n : nat => hzremaindermod m x (a n)))) _ _).
rewrite <- (hzquotientmodandplus). apply idpath. Defined.

```

```

Lemma hzquotientandtimesl (m : hz) (x : hzneq 0 m) (a b : hz) :
hzquotientmod m x (a * b) ~> ((hzquotientmod m x a) * b + (
hzremaindermod m x a) * (hzquotientmod m x b) + hzquotientmod m x (
(hzremaindermod m x a) * (hzremaindermod m x b))). Proof.
intros. rewrite hzquotientmodandtimes. rewrite (hzmultcomm (
hzremaindermod m x b) (hzquotientmod m x a)). rewrite
hzmultassoc. rewrite <- (hzldistr (hzquotientmod m x b * m) _ (
hzquotientmod m x a)). rewrite (hzmultcomm _ m). rewrite <- (
hzdivequationmod m x b). rewrite hzplusassoc. apply idpath.
Defined.

```

```

Lemma hzquotientandfpstimesl (m : hz) (x : hzneq 0 m) (a b : nat
-> hz) (upper : nat) : hzquotientmod m x (fpstimes hz a b upper)
~> (natsummation0 upper (fun i : nat => (hzquotientmod m x (a i)
) * b (minus upper i)) + hzquotientmod m x (natsummation0 upper (
fun i : nat => (hzremaindermod m x (a i)) * b (minus upper i))
)). Proof. intros. destruct upper. simpl. unfold
fpstimes. simpl. rewrite hzquotientandtimesl. rewrite hzplusassoc.
apply (maponpaths (fun v : _ => hzquotientmod m x (a 0%nat) * b
0%nat + v)). rewrite (hzquotientmodandtimes m x (hzremaindermod m
x (a 0%nat)) (b 0%nat)). rewrite <- hzgrandremainderq. rewrite
hzmultx0. rewrite 2! hzmult0x. rewrite hzplusr0. rewrite
hzremaindermoditerated. apply idpath. unfold fpstimes. rewrite
hzgrandnatsummation0q. assert (forall n : nat, hzquotientmod m x (a n
* b (minus (S upper) n)%nat) ~> ((hzquotientmod m x (a n)) * b
(minus (S upper) n) + (hzremaindermod m x (a n)) * (
hzquotientmod m x (b (minus (S upper) n))) + hzquotientmod m x
((hzremaindermod m x (a n)) * (hzremaindermod m x (b (minus (
S upper) n)))))) as f. intro k. rewrite hzquotientandtimesl.
apply idpath. rewrite (natsummationpathsupperfixed _ _ (fun x0 p =>
f x0)). rewrite (natsummationplustidistr (S upper) (fun x0 : nat
=> hzquotientmod m x (a x0) * b (minus (S upper) x0)%nat +
hzremaindermod m x (a x0) * hzquotientmod m x (b (S upper - x0)%nat)
)). rewrite (natsummationplustidistr (S upper) (fun x0 : nat =>
hzquotientmod m x (a x0) * b (S upper - x0)%nat)). rewrite 2!
hzplusassoc. apply (maponpaths (fun v : _ => natsummation0 (S upper)
) (fun i : nat => hzquotientmod m x (a i) * b (minus (S upper) i)
)) + v)). rewrite (hzgrandnatsummation0q m x (fun i : nat =>
hzremaindermod m x (a i) * b (minus (S upper) i))). assert (
(natsummation0 (S upper) (fun n : nat => hzremaindermod m x
(hzremaindermod m x (a n) * b (S upper - n)%nat))) ~> (natsummation0
(S upper) (fun n : nat => hzremaindermod m x (a n * b (minus (S
upper) n)))))) as g. apply natsummationpathsupperfixed. intros j
p. rewrite hzremaindermodandtimes. rewrite
hzremaindermoditerated. rewrite <- hzremaindermodandtimes. apply
idpath. rewrite g. rewrite <- hzplusassoc. assert (natsummation0 (S
upper) (fun x0 : nat => hzremaindermod m x (a x0) * hzquotientmod m x
(b (S upper - x0)%nat) + natsummation0 (S upper) (fun x0 : nat =>
hzquotientmod m x (hzremaindermod m x (a x0) * hzremaindermod m x (b
(S upper - x0)%nat))) ~> natsummation0 (S upper) (fun n : nat =>
hzquotientmod m x (hzremaindermod m x (a n) * b (S upper - n)%nat)))
as h. rewrite <- (natsummationplustidistr (S upper) (fun x0 : nat =>
hzremaindermod m x (a x0) * hzquotientmod m x (b (minus (S upper)
x0))))). apply natsummationpathsupperfixed. intros j p. rewrite (
hzquotientmodandtimes m x (hzremaindermod m x (a j)) (b (minus
(S upper) j))))). rewrite <- hzgrandremainderq. rewrite 2!
hzmult0x. rewrite hzmultx0. rewrite hzplusr0. rewrite
hzremaindermoditerated. apply idpath. rewrite h. apply idpath.
Defined.

```

```

Close Scope hz_scope.

```

(** * II. The carrying operation and induced equivalence relation on formal power series *)

Open Scope rng_scope.

Fixpoint precarry (m : hz) (is : hzneq 0 m) (a : fpscommrng hz)
 (n : nat) : hz := match n with | 0%nat => a 0%nat | S n => a (S n)
 + (hzquotientmod m is (precarry m is a n)) end.

Definition carry (m : hz) (is : hzneq 0 m) : fpscommrng hz ->
 fpscommrng hz := fun a : fpscommrng hz => fun n : nat =>
 hzremaindermod m is (precarry m is a n).

(* precarry and carry are as described in the following example:

CASE: mod 3

First we normalize the sequence as we go along:

5 6 8 4 (13) 2 2 (remainder 2 mod 3 = 2) 4 1 (remainder 13 mod 3
 = 1, quotient 13 mod 3 = 4) 2 2 (remainder 8 mod 3 =
 2, quotient 8 mod 3 = 2) 3 1 (remainder 10 mod 3 = 1,
 quotient 10 mod 3 = 3) 3 0 (remainder 9 mod 3 = 0,
 quotient 9 mod 3 = 3) 2 2 (remainder 8 mod 3 = 2,
 quotient 8 mod 3 = 2)

2 2 0 1 2 1 2

Next we first precarry and then carry:

5 6 8 4 (13) 2 2 4 13 2 8 3 (10) 3 9 2 8

2 8 9 (10) 8 (13) 2 <--- precarried sequence

2 2 0 1 2 1 2 <--- carried sequence *)

Lemma isapropcarryequiv (m : hz) (is : hzneq 0 m) (a b :
 fpscommrng hz) : isaprop ((carry m is a) ^> (carry m is b)).
 Proof. intros. apply (fps hz). Defined.

Definition carryequiv0 (m : hz) (is : hzneq 0 m) : hrel (
 fpscommrng hz) := fun a b : fpscommrng hz => hProppair _ (
 isapropcarryequiv m is a b).

Lemma carryequiviseqrel (m : hz) (is : hzneq 0 m) : iseqrel (
 carryequiv0 m is). Proof. intros. split. split. intros a b c i
 j. simpl. rewrite i. apply j. intros a. simpl. apply idpath. intros a
 b i. simpl. rewrite i. apply idpath. Defined.

Lemma carryandremainder (m : hz) (is : hzneq 0 m) (a : fpscommrng
 hz) (n : nat) : hzremaindermod m is (carry m is a n) ^> carry m
 is a n. Proof. intros. unfold carry. rewrite
 hzremaindermoditerated. apply idpath. Defined.

Definition carryequiv (m : hz) (is : hzneq 0 m) : eqrel (
 fpscommrng hz) := eqrelpair _ (carryequiviseqrel m is).

Lemma precarryandcarry (m : hz) (is : hzneq 0 m) (a : fpscommrng
 hz) : precarry m is (carry m is a) ^> carry m is a. Proof.
 intros. assert (forall n : nat, (precarry m is (carry m is a)) n
 ^> ((carry m is a) n)) as f. intros n. induction n. simpl. apply
 idpath. simpl. rewrite lHn. unfold carry at 2. rewrite <-
 hzgrandremainderq. rewrite hzplusr0. apply idpath. apply (funextfun _
 _ f). Defined.

Lemma hzgrandcarryeq (m : hz) (is : hzneq 0 m) (a : fpscommrng hz

) (n : nat) : carry m is a n ^> ((m * 0) + carry m is a n).
 Proof. intros. rewrite hzmultx0. rewrite hzplusl0. apply idpath.
 Defined.

Lemma hzgrandcarryineq (m : hz) (is : hzneq 0 m) (a : fpscommrng
 hz) (n : nat) : dirprod (hzleh 0 (carry m is a n)) (hzlt (
 carry m is a n) (nattohz (hzabsval m))). Proof.
 intros. split. unfold carry. apply (pr2 (pr1 (divalgorithm (
 precarry m is a n) m is))). unfold carry. apply (pr2 (pr1 (
 divalgorithm (precarry m is a n) m is))). Defined.

Lemma hzgrandcarryq (m : hz) (is : hzneq 0 m) (a : fpscommrng hz
) (n : nat) : 0 ^> hzquotientmod m is (carry m is a n). Proof.
 intros. apply (hzqrstestq m is (carry m is a n) 0 (carry m is a n)
). split. apply hzgrandcarryeq. apply hzgrandcarryineq. Defined.

Lemma hzgrandcarryr (m : hz) (is : hzneq 0 m) (a : fpscommrng hz
) (n : nat) : carry m is a n ^> hzremaindermod m is (carry m is a n
). Proof. intros. apply (hzqrstestr m is (carry m is a n) 0 (
 carry m is a n)). split. apply hzgrandcarryeq. apply
 hzgrandcarryineq. Defined.

Lemma doublecarry (m : hz) (is : hzneq 0 m) (a : fpscommrng hz)
 : carry m is (carry m is a) ^> carry m is a. Proof. intros. assert
 (forall n : nat, (carry m is (carry m is a)) n ^> ((carry m is
 a) n)) as f. intros. induction n. unfold carry. simpl. apply
 hzremaindermoditerated. unfold carry. simpl. change (precarry m is
 (fun n0 : nat => hzremaindermod m is (precarry m is a n0)) n) with ((
 precarry m is (carry m is a)) n). rewrite
 precarryandcarry. rewrite <- hzgrandcarryq. rewrite hzplusr0. rewrite
 hzremaindermoditerated. apply idpath. apply (funextfun _ _ f).
 Defined.

Lemma carryandcarryequiv (m : hz) (is : hzneq 0 m) (a :
 fpscommrng hz) : carryequiv m is (carry m is a) a. Proof.
 intros. simpl. rewrite doublecarry. apply idpath. Defined.

Lemma quotientprecarryplus (m : hz) (is : hzneq 0 m) (a b :
 fpscommrng hz) (n : nat) : hzquotientmod m is (precarry m is (a +
 b) n) ^> (hzquotientmod m is (precarry m is a n) + hzquotientmod
 m is (precarry m is b n) + hzquotientmod m is (precarry m is (
 carry m is a + carry m is b) n)). Proof. intros. induction
 n. simpl. change (hzquotientmod m is (a 0%nat + b 0%nat) ^>
 (hzquotientmod m is (a 0%nat) + hzquotientmod m is (b 0%nat) +
 hzquotientmod m is (hzremaindermod m is (a 0%nat) + hzremaindermod
 m is (b 0%nat)))). rewrite hzquotientmodandplus. apply idpath.

change (hzquotientmod m is (a (S n) + b (S n) + hzquotientmod
 m is (precarry m is (a + b) n)) ^> (hzquotientmod m is (precarry
 m is a (S n)) + hzquotientmod m is (precarry m is b (S n)) +
 hzquotientmod m is (carry m is a (S n) + carry m is b (S n) +
 hzquotientmod m is (precarry m is (carry m is a + carry m is b) n))
)). rewrite lHn. rewrite (rngassoc1 hz (a (S n)) (b (S n)
) _). rewrite <- (rngassoc1 hz (b (S n))). rewrite (rngcomm1
 hz (b (S n)) _). rewrite <- 3! (rngassoc1 hz (a (S n)) _
 _). change (a (S n) + hzquotientmod m is (precarry m is a n)
) with (precarry m is a (S n)). set (pa := precarry m is a (S
 n)). rewrite (rngassoc1 hz pa _ (b (S n))). rewrite (
 rngcomm1 hz _ (b (S n))). change (b (S n) + hzquotientmod m
 is (precarry m is b n)) with (precarry m is b (S n)). set (
 pb := precarry m is b (S n)). set (ab := precarry m is (carry
 m is a + carry m is b)). rewrite (rngassoc1 hz (carry m is a (
 S n)) (carry m is b (S n)) (hzquotientmod m is (ab n))).
 rewrite (hzquotientmodandplus m is (carry m is a (S n)) _).
 unfold carry at 1. rewrite <- hzgrandremainderq. rewrite hzplusl0.
 rewrite (hzquotientmodandplus m is (carry m is b (S n)) _).

```

unfold carry at 1. rewrite <- hzgrandremainderq. rewrite hzplusl0.
rewrite (rngassoc1 hz pa pb _). rewrite (hzquotientmodandplus m
is pa _). change (pb + hzquotientmod m is (ab n)) with (pb +
hzquotientmod m is (ab n))%hz. rewrite (hzquotientmodandplus m is
pb (hzquotientmod m is (ab n)) _). rewrite <- 2! (rngassoc1 hz
(hzquotientmod m is pa) _ _). rewrite <- 2! (rngassoc1 hz
(hzquotientmod m is pa + hzquotientmod m is pb) _). rewrite 2! (
rngassoc1 hz (hzquotientmod m is pa + hzquotientmod m is pb +
hzquotientmod m is (hzquotientmod m is (ab n)) _ _). apply (
maponpaths (fun x : hz => (hzquotientmod m is pa + hzquotientmod m
is pb + hzquotientmod m is (hzquotientmod m is (ab n)) _ + x))).
unfold carry at 1 2. rewrite 2! hzremaindermoditerated. change (
precarry m is b (S n)) with pb. change (precarry m is a (S n)
) with pa. apply (maponpaths (fun x : hz => (hzquotientmod m is
(hzremaindermod m is pb + hzremaindermod m is (hzquotientmod m is
(ab n))%hz) + x))). apply maponpaths. apply (maponpaths (fun x
: hz => hzremaindermod m is pa + x))). rewrite (
hzremaindermodandplus m is (carry m is b (S n)) _). unfold
carry. rewrite hzremaindermoditerated. rewrite <- (
hzremaindermodandplus m is (precarry m is b (S n)) _). apply
idpath. Defined.

```

```

Lemma carryandplus (m : hz) (is : hzneq 0 m) (a b : fpscommrng hz) :
carry m is (a + b) ^> carry m is (carry m is a + carry m is b).
Proof. intros. assert (forall n : nat, carry m is (a + b) n ^>
(carry m is (carry m is a + carry m is b) n)) as f. intros
n. destruct n. change (hzremaindermod m is (a 0%nat + b 0%nat) ^>
hzremaindermod m is (hzremaindermod m is (a 0%nat) + hzremaindermod
m is (b 0%nat))) . rewrite hzremaindermodandplus. apply idpath.
change (hzremaindermod m is (a (S n) + b (S n) + hzquotientmod m
is (precarry m is (a + b) n)) ^> hzremaindermod m is (
hzremaindermod m is (a (S n) + hzquotientmod m is (precarry m is a
n)) + hzremaindermod m is (b (S n) + hzquotientmod m is (
precarry m is b n)) + hzquotientmod m is (precarry m is (carry m
is a + carry m is b) n))). rewrite quotientprecarryplus. rewrite
(hzremaindermodandplus m is (hzremaindermod m is (a (S n) +
hzquotientmod m is (precarry m is a n)) + hzremaindermod m is (b (S n)
+ hzquotientmod m is (precarry m is b n)))). change
(hzremaindermod m is (a (S n) + hzquotientmod m is (precarry m is a
n)) + hzremaindermod m is (b (S n) + hzquotientmod m is (precarry m
is b n))) with (hzremaindermod m is (a (S n) + hzquotientmod m is
(precarry m is a n))%rng + hzremaindermod m is (b (S n) +
hzquotientmod m is (precarry m is b n))%rng)%hz. rewrite <-
(hzremaindermodandplus m is (a (S n) + hzquotientmod m is (precarry m
is a n)) (b (S n) + hzquotientmod m is (precarry m is b n))).
rewrite <- hzremaindermodandplus. change ((a (S n) + hzquotientmod
m is (precarry m is a n))%rng + (b (S n) + hzquotientmod m is
(precarry m is b n))%rng + hzquotientmod m is (precarry m is (carry m
is a + carry m is b)%rng n))%hz) with ((a (S n) + hzquotientmod m
is (precarry m is a n))%rng + (b (S n) + hzquotientmod m is (precarry m
is b n))%rng + hzquotientmod m is (precarry m is (carry m is a + carry
m is b)%rng n))%rng. rewrite <- (rngassoc1 hz (a (S n) +
hzquotientmod m is (precarry m is a n)) (b (S n)) (hzquotientmod
m is (precarry m is b n))). rewrite (rngassoc1 hz (a (S n))
(hzquotientmod m is (precarry m is a n)) (b (S n))). rewrite (
rngcomm1 hz (hzquotientmod m is (precarry m is a n)) (b (S n))
). rewrite <- 3! (rngassoc1 hz). apply idpath. apply (funextfun _
_ f). Defined.

```

```

Definition quotientprecarry (m : hz) (is : hzneq 0 m) (a :
fpscommrng hz) : fpscommrng hz := fun x : nat => hzquotientmod m is (
precarry m is a x).

```

```

Lemma quotientandtimesrearrangel (m : hz) (is : hzneq 0 m) (x y :
hz) : hzquotientmod m is (x * y) ^> ((hzquotientmod m is x) * y
+ hzquotientmod m is ((hzremaindermod m is x) * y)). Proof.

```

```

intros. rewrite hzquotientmodandtimes. change (hzquotientmod m is x *
hzquotientmod m is y * m + hzremaindermod m is y * hzquotientmod m is
x + hzremaindermod m is x * hzquotientmod m is y + hzquotientmod m is
(hzremaindermod m is x * hzremaindermod m is y))%hz with
(hzquotientmod m is x * hzquotientmod m is y * m + hzremaindermod m is
y * hzquotientmod m is x + hzremaindermod m is x * hzquotientmod m is
y + hzquotientmod m is (hzremaindermod m is x * hzremaindermod m is
y))%rng. rewrite (rngcomm2 hz (hzremaindermod m is y) (
hzquotientmod m is x)). rewrite (rngassoc2 hz). rewrite <- (
rngldistr hz). rewrite (rngcomm2 hz (hzquotientmod m is y) m).
change (m * hzquotientmod m is y + hzremaindermod m is y)%rng with (m
* hzquotientmod m is y + hzremaindermod m is y)%hz. rewrite <- (
hzdivequationmod m is y). change (hzremaindermod m is x * y)%rng with
(hzremaindermod m is x * y)%hz. rewrite (hzquotientmodandtimes m is
(hzremaindermod m is x) y). rewrite
hzremaindermoditerated. rewrite <- hzgrandremainderq. rewrite
hzmultx0. rewrite 2! hzmult0x. rewrite hzplusl0. rewrite (rngassoc1
hz). change (hzquotientmod m is x * y + (hzremaindermod m is x *
hzquotientmod m is y + hzquotientmod m is (hzremaindermod m is x *
hzremaindermod m is y))%hz) with (hzquotientmod m is x * y +
(hzremaindermod m is x * hzquotientmod m is y + hzquotientmod m is
(hzremaindermod m is x * hzremaindermod m is y))%rng. apply idpath.
Defined.

```

```

Lemma natsummationplusshift {R : commrng} (upper : nat) (f g :
nat -> R) : (natsummation0 (S upper) f) + (natsummation0 upper g)
^> (f 0%nat + (natsummation0 upper (fun x : nat => f (S x) + g
x))) . Proof. intros. destruct upper. unfold
natsummation0. simpl. apply (rngassoc1 R). rewrite (
natsummationshift0 (S upper) f). rewrite (rngcomm1 R _ (f 0%nat)
). rewrite (rngassoc1 R). rewrite natsummationplusdistr. apply
idpath. Defined.

```

```

Lemma precarryandtimesl (m : hz) (is : hzneq 0 m) (a b :
fpscommrng hz) (n : nat) : hzquotientmod m is (precarry m is (a *
b) n) ^> ((quotientprecarry m is a * b) n + hzquotientmod m is (
precarry m is ((carry m is a) * b) n)). Proof.
intros. induction n. unfold precarry. change ((a * b) 0%nat) with
(a 0%nat * b 0%nat). change ((quotientprecarry m is a * b) 0%nat)
with (hzquotientmod m is (a 0%nat) * b 0%nat). rewrite
quotientandtimesrearrangel. change ((carry m is a * b) 0%nat)
with (hzremaindermod m is (a 0%nat) * b 0%nat). apply idpath.

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change (precarry m is (a * b) (S n)) with ((a * b) (S n)
+ hzquotientmod m is (precarry m is (a * b) n)). rewrite
IHn. rewrite <- (rngassoc1 hz). assert (((a * b) (S n)) + (
quotientprecarry m is a * b) n) ^> (@op2 (fpscommrng hz) (
precarry m is a) b) (S n)) as f. change ((a * b) (S n))
with (natsummation0 (S n) (fun x : nat => a x * b (minus (S n)
x))). change ((quotientprecarry m is a * b) n) with (
natsummation0 n (fun x : nat => quotientprecarry m is a x * b (
minus n x))). rewrite natsummationplusshift. change ((@op2 (
fpscommrng hz) (precarry m is a) b) (S n)) with (
natsummation0 (S n) (fun x : nat => (precarry m is a) x * b (
minus (S n) x))). rewrite natsummationshift0. unfold precarry
at 2. simpl. rewrite <- (rngcomm1 hz (a 0%nat * b (S n)) _
). apply (maponpaths (fun x : hz => a 0%nat * b (S n) + x)
). apply natsummationpathsupperfixed. intros k j. unfold
quotientprecarry. rewrite (rngldistr hz). apply idpath. rewrite
f. rewrite hzquotientmodandplus. change (@op2 (fpscommrng hz) (
precarry m is a) b) with (fpstimes hz (precarry m is a) b)
). rewrite (hzquotientandfpstimesl m is (precarry m is a) b)
). change (@op2 (fpscommrng hz) (carry m is a) b) with (
fpstimes hz (carry m is a) b) at 1. unfold fpstimes at 1. unfold
carry at 1. change (fun n0 : nat => let t' := fun m0 : nat => b (n0
- m0)%nat in natsummation0 n0 (fun x : nat => (hzremaindermod m is

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(hzquotientmod m is (precary m is a x) * t' x)%rng)) with (carry m is a * b). change
((quotientprecary m is a * b) (S n)) with (natsummation0 (S n) (fun i : nat => hzquotientmod m is (precary m is a i) * b (S n - i)%nat)). rewrite 2! hzplusassoc. apply (maponpaths (fun v : _ => natsummation0 (S n) (fun i : nat => hzquotientmod m is (precary m is a i) * b (S n - i)%nat) + v)). change (precary m is (carry m is a * b) (S n)) with ((carry m is a * b) (S n) + hzquotientmod m is (precary m is (carry m is a * b) n)). change ((carry m is a * b) (S n) + hzquotientmod m is (precary m is (carry m is a * b) n)) with ((carry m is a * b)%rng (S n) + hzquotientmod m is (precary m is (carry m is a * b) n)%rng)%hz. rewrite (hzquotientmodandplus m is ((carry m is a * b) (S n)) (hzquotientmod m is (precary m is (carry m is a * b) n))). change ((carry m is a * b) (S n)) with (natsummation0 (S n) (fun i : nat => hzremaindermod m is (precary m is a i) * b (S n - i)%nat)). rewrite hzplusassoc. apply (maponpaths (fun v : _ => (hzquotientmod m is (natsummation0 (S n) (fun i : nat => hzremaindermod m is (precary m is a i) * b (S n - i)%nat) + v)). apply (maponpaths (fun v : _ => hzquotientmod m is (hzquotientmod m is (precary m is (carry m is a * b)%rng n) + v)). apply maponpaths. apply (maponpaths (fun v : _ => v + hzremaindermod m is (hzquotientmod m is (precary m is (carry m is a * b)%rng n)))). unfold fpstimes. rewrite hzgrandnatsummation0r. rewrite (hzgrandnatsummation0r m is (fun i : nat => hzremaindermod m is (precary m is a i) * b (S n - i)%nat)). apply maponpaths. apply natsummationpathsupperfixed. intros j p. change (hzremaindermod m is (hzremaindermod m is (precary m is a j) * b (minus (S n) j)) with (hzremaindermod m is (hzremaindermod m is (precary m is a j) * b (S n - j)%nat)%hz). rewrite (hzremaindermodandtimes m is (hzremaindermod m is (precary m is a j)) (b (minus (S n) j))). rewrite hzremaindermoditerated. rewrite <- hzremaindermodandtimes. apply idpath. Defined.

Lemma carryandtimesl (m : hz) (is : hzneq 0 m) (a b : fpscommrng hz) : carry m is (a * b) -> carry m is (carry m is a * b). Proof. intros. assert (forall n : nat, carry m is (a * b) n -> carry m is (carry m is a * b) n) as f. intros n. destruct n. unfold carry at 1 2. change (precary m is (a * b) 0%nat) with (a 0%nat * b 0%nat). change (precary m is (carry m is a * b) 0%nat) with (carry m is a 0%nat * b 0%nat). unfold carry. change (hzremaindermod m is (precary m is a 0) * b 0%nat) with (hzremaindermod m is (precary m is a 0) * b 0%nat)%hz. rewrite (hzremaindermodandtimes m is (hzremaindermod m is (precary m is a 0%nat)) (b 0%nat)). rewrite hzremaindermoditerated. rewrite <- hzremaindermodandtimes. change (precary m is a 0%nat) with (a 0%nat). apply idpath. unfold carry at 1 2. change (precary m is (a * b) (S n)) with ((a * b) (S n) + hzquotientmod m is (precary m is (a * b) n)). rewrite precaryandtimesl. rewrite <- (rngassoc1 hz). rewrite hzremaindermodandplus. assert (hzremaindermod m is ((a * b) (S n) + (quotientprecary m is a * b) n) -> hzremaindermod m is ((carry m is a * b) (S n))) as g. change (hzremaindermod m is ((natsummation0 (S n) (fun u : nat => a u * b (minus (S n) u))) + (natsummation0 n (fun u : nat => (quotientprecary m is a) u * b (minus n u)))) -> hzremaindermod m is (natsummation0 (S n) (fun u : nat => (carry m is a) u * b (minus (S n) u)))). rewrite (natsummationplusshift n). rewrite (natsummationshift0 n (fun u : nat => carry m is a u * b (minus (S n) u))). assert (hzremaindermod m is (natsummation0 n (fun x : nat => a (S x) * b (minus (S n) (S x)) + quotientprecary m is a * b (minus n x))) -> hzremaindermod m is (natsummation0 n (fun x : nat => carry m is a (S x) * b (minus (S n) (S x))))) as h. rewrite hzgrandnatsummation0r. rewrite (hzgrandnatsummation0r m is (fun x : nat => carry m is a (S x) * b (minus (S n) (S x)))). apply maponpaths. apply natsummationpathsupperfixed. intros j p. unfold quotientprecary. simpl. change (a (S j) * b (minus n j) +

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hzquotientmod m is (precary m is a j) * b (minus n j)) with (a (S j) * b (minus n j) + hzquotientmod m is (precary m is a j) * b (minus n j))%hz. rewrite <- (hzrdistr (a (S j)) (hzquotientmod m is (precary m is a j)) (b (minus n j))). rewrite hzremaindermodandtimes. change (hzremaindermod m is (hzremaindermod m is (a (S j) + hzquotientmod m is (precary m is a j)) * hzremaindermod m is (b (minus n j))) -> hzremaindermod m is (carry m is a (S j) * b (minus n j))%rng. rewrite <- (hzremaindermoditerated m is (a (S j) + hzquotientmod m is (precary m is a j))). unfold carry. rewrite <- hzremaindermodandtimes. apply idpath. rewrite hzremaindermodandplus. rewrite h. rewrite <- hzremaindermodandplus. unfold carry at 3. rewrite (hzremaindermodandplus m is _ (hzremaindermod m is (precary m is a 0%nat) * b (minus (S n) 0%nat))). rewrite hzremaindermodandtimes. rewrite hzremaindermoditerated. rewrite <- hzremaindermodandtimes. change (precary m is a 0%nat) with (a 0%nat). rewrite <- hzremaindermodandplus. rewrite hzpluscomm. apply idpath. rewrite g. rewrite <- hzremaindermodandplus. apply idpath. apply (funextfun _ f). Defined.

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Lemma carryandtimesr (m : hz) (is : hzneq 0 m) (a b : fpscommrng hz) : carry m is (a * b) -> carry m is (a * carry m is b). Proof. intros. rewrite (@rngcomm2 (fpscommrng hz)). rewrite carryandtimesl. rewrite (@rngcomm2 (fpscommrng hz)). apply idpath. Defined.

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Lemma carryandtimes (m : hz) (is : hzneq 0 m) (a b : fpscommrng hz) : carry m is (a * b) -> carry m is (carry m is a * carry m is b). Proof. intros. rewrite carryandtimesl. rewrite carryandtimesr. apply idpath. Defined.

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Lemma rngcarryequiv (m : hz) (is : hzneq 0 m) : @rngqrel (fpscommrng hz). Proof. intros. split with (carryequiv m is). split. split. intros a b c q. simpl. simpl in q. rewrite carryandplus. rewrite q. rewrite <- carryandplus. apply idpath. intros a b c q. simpl. rewrite carryandplus. rewrite q. rewrite <- carryandplus. apply idpath. split. intros a b c q. simpl. rewrite carryandtimes. rewrite q. rewrite <- carryandtimes. apply idpath. intros a b c q. simpl. rewrite carryandtimes. rewrite q. rewrite <- carryandtimes. apply idpath. Defined.

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Definition commrngofadicints (p : hz) (is : isapime p) := commrngquot (rngcarryequiv p (isaprimetoneq0 is)).

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Definition padicplus (p : hz) (is : isapime p) := @op1 (commrngofadicints p is).

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Definition padictimes (p : hz) (is : isapime p) := @op2 (commrngofadicints p is).

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(** III. The apartness relation on p-adic integers *)

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Definition padicapart0 (p : hz) (is : isapime p) : hrel (fpscommrng hz) := fun a b : _ => (hexists (fun n : nat => (neq _ (carry p (isaprimetoneq0 is) a n) (carry p (isaprimetoneq0 is) b n))))).

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Lemma padicapartiscomprel (p : hz) (is : isapime p) : iscomprelrel (carryequiv p (isaprimetoneq0 is)) (padicapart0 p is). Proof. intros p is a' b' i j. apply uahp. intro k. apply k. intros u. destruct u as [n u]. apply total2tohexists. split with n. rewrite <- i, <- j. assumption. intro k. apply k. intros u. destruct u as [n u]. apply total2tohexists. split with n. rewrite i, j. assumption. Defined.

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Definition padicapart1 (p : hz) (is : isapime p) : hrel (

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`commrngofpadicints p is) := quotrel (padicapartiscomprel p is).`

Lemma `isirreflpadicapart0 (p : hz) (is : isapime p) : isirrefl (padicapart0 p is).` Proof. intros. intros a f. simpl in f. assert hfals as x. apply f. intros u. destruct u as [n u]. apply u. apply idpath. apply x. Defined.

Lemma `issympadicapart0 (p : hz) (is : isapime p) : issymm (padicapart0 p is).` Proof. intros. intros a b f. apply f. intros u. destruct u as [n u]. apply total2tohexists. split with n. intros g. apply u. rewrite g. apply idpath. Defined.

Lemma `iscotranspadicapart0 (p : hz) (is : isapime p) : iscotrans (padicapart0 p is).` Proof. intros. intros a b c f. apply f. intros u. destruct u as [n u]. intros P j. apply j. destruct (isdeceqhz (carry p (isaprimetoneq0 is) a n) (carry p (isaprimetoneq0 is) b n)) as [l | r]. apply ii2. intros Q k. apply k. split with n. intros g. apply u. rewrite l, g. apply idpath. apply iiii. intros Q k. apply k. split with n. intros g. apply r. assumption. Defined.

Definition `padicapart (p : hz) (is : isapime p) : apart (commrngofpadicints p is).` Proof. intros. split with (padicapart1 p is). split. unfold padicapart1. apply (isirreflquotrel (padicapartiscomprel p is) (isirreflpadicapart0 p is)). split. apply (issymmquotrel (padicapartiscomprel p is) (issympadicapart0 p is)). apply (iscotransquotrel (padicapartiscomprel p is) (iscotranspadicapart0 p is)). Defined.

Lemma `precaryandzero (p : hz) (is : isapime p) : precarry p (isaprimetoneq0 is) 0 ~> (@rngunel1 (fpscommrng hz)).` Proof. intros. assert (forall n : nat, precarry p (isaprimetoneq0 is) 0 n ~> (@rngunel1 (fpscommrng hz)) n) as f. intros n. induction n. unfold precarry. change ((@rngunel1 (fpscommrng hz)) 0%nat) with 0%hz. apply idpath. change ((@rngunel1 (fpscommrng hz) (S n)) + hzquotientmod p (isaprimetoneq0 is) (precarry p (isaprimetoneq0 is) (@rngunel1 (fpscommrng hz)) n)) ~> 0%hz. rewrite IHn. change ((@rngunel1 (fpscommrng hz)) n) with 0%hz. change ((@rngunel1 (fpscommrng hz)) (S n)) with 0%hz. rewrite hzqrand0q. rewrite hzplusl0. apply idpath. apply (funextfun _ _ f). Defined.

Lemma `carryandzero (p : hz) (is : isapime p) : carry p (isaprimetoneq0 is) 0 ~> 0.` Proof. intros. unfold carry. rewrite precarryandzero. assert (forall n : nat, (fun n : nat => hzremaindermod p (isaprimetoneq0 is) ((@rngunel1 (fpscommrng hz)) n)) n ~> (@rngunel1 (fpscommrng hz)) n) as f. intros n. rewrite hzqrand0r. unfold carry. change ((@rngunel1 (fpscommrng hz)) n) with 0%hz. apply idpath. apply (funextfun _ _ f). Defined.

Lemma `precaryandone (p : hz) (is : isapime p) : precarry p (isaprimetoneq0 is) 1 ~> (@rngunel2 (fpscommrng hz)).` Proof. intros. assert (forall n : nat, precarry p (isaprimetoneq0 is) 1 n ~> (@rngunel2 (fpscommrng hz)) n) as f. intros n. induction n. unfold precarry. apply idpath. simpl. rewrite IHn. destruct n. change ((@rngunel2 (fpscommrng hz)) 0%nat) with 1%hz. rewrite hzqrandiq. rewrite hzplusr0. apply idpath. change ((@rngunel2 (fpscommrng hz)) (S n)) with 0%hz. rewrite hzqrand0q. rewrite hzplusr0. apply idpath. apply (funextfun _ _ f). Defined.

Lemma `carryandone (p : hz) (is : isapime p) : carry p (isaprimetoneq0 is) 1 ~> 1.` Proof. intros. unfold carry. rewrite precaryandone. assert (forall n : nat, (fun n : nat => hzremaindermod p (isaprimetoneq0 is) ((@rngunel2 (fpscommrng hz)) n)) n ~> (@rngunel2 (fpscommrng hz)) n) as f. intros n. destruct n. change ((@rngunel2 (fpscommrng hz)) 0%nat) with 1%hz. rewrite hzqrandir. apply idpath. change ((@rngunel2 (

`fpscommrng hz)) (S n)) with 0%hz. rewrite hzqrand0r. apply idpath. apply (funextfun _ _ f). Defined.`

Lemma `padicapartcomputation (p : hz) (is : isapime p) (a b : fpscommrng hz) : (pr1 (padicapart p is)) (setquotpr (carryequiv p (isaprimetoneq0 is)) a) (setquotpr (carryequiv p (isaprimetoneq0 is)) b) ~> padicapart0 p is a b.` Proof. intros. apply uahp. intros i. apply i. intro u. apply u. Defined.

Lemma `padicapartandplusprecaryl (p : hz) (is : isapime p) (a b c : fpscommrng hz) (n : nat) (x : neq _ (precarry p (isaprimetoneq0 is) (carry p (isaprimetoneq0 is) a + carry p (isaprimetoneq0 is) b) n) ((precarry p (isaprimetoneq0 is) (carry p (isaprimetoneq0 is) a + carry p (isaprimetoneq0 is) c)) n)) : (padicapart0 p is) b c.` Proof. intros. set (P := fun x : nat => neq hz (precarry p (isaprimetoneq0 is) (carry p (isaprimetoneq0 is) a + carry p (isaprimetoneq0 is) b) x) (precarry p (isaprimetoneq0 is) (carry p (isaprimetoneq0 is) a + carry p (isaprimetoneq0 is) c) x)). assert (isdecnatprop P) as isdec. intros m. destruct (isdeceqhz (precarry p (isaprimetoneq0 is) (carry p (isaprimetoneq0 is) a + carry p (isaprimetoneq0 is) b) m) (precarry p (isaprimetoneq0 is) (carry p (isaprimetoneq0 is) a + carry p (isaprimetoneq0 is) c) m)) as [l | r]. apply ii2. intros j. apply j. assumption. apply iiii. assumption. set (leexists := leastelementprinciple n P isdec x). apply leexists. intro k. destruct k as [k k']. destruct k' as [k' k'']. destruct k. apply total2tohexists. split with 0%nat. intros i. apply k'. change (carry p (isaprimetoneq0 is) a 0%nat + carry p (isaprimetoneq0 is) b 0%nat ~> (carry p (isaprimetoneq0 is) a 0%nat + carry p (isaprimetoneq0 is) c 0%nat)). rewrite i. apply idpath. apply total2tohexists. split with (S k). intro i. apply (k' k'). apply natlthnsn. intro j. apply k'. change (carry p (isaprimetoneq0 is) a (S k) + carry p (isaprimetoneq0 is) b (S k)) + hzquotientmod p (isaprimetoneq0 is) (precarry p (isaprimetoneq0 is) (carry p (isaprimetoneq0 is) a + carry p (isaprimetoneq0 is) b) k) ~> (carry p (isaprimetoneq0 is) a (S k) + carry p (isaprimetoneq0 is) c (S k) + hzquotientmod p (isaprimetoneq0 is) (precarry p (isaprimetoneq0 is) (carry p (isaprimetoneq0 is) a + carry p (isaprimetoneq0 is) c) k))). rewrite i. rewrite j. apply idpath. Defined.

Lemma `padicapartandplusprecaryr (p : hz) (is : isapime p) (a b c : fpscommrng hz) (n : nat) (x : neq _ (precarry p (isaprimetoneq0 is) (carry p (isaprimetoneq0 is) b + carry p (isaprimetoneq0 is) a) n) ((precarry p (isaprimetoneq0 is) (carry p (isaprimetoneq0 is) c + carry p (isaprimetoneq0 is) a)) n)) : (padicapart0 p is) b c.` Proof. intros. rewrite 2! (rngcomm1 (fpscommrng hz) _ (carry p (isaprimetoneq0 is) a)) in x. apply (padicapartandplusprecaryl p is a b c n x). Defined.

Lemma `commrngquotrandop1 { A : commrng } (R : @rngeqrel A) (a b : A) : (@op1 (commrngquot R)) (setquotpr (pr1 R) a) (setquotpr (pr1 R) b) ~> setquotpr (pr1 R) (a + b).` Proof. intros. change (@op1 (commrngquot R)) with (setquotfun2 R R (@op1 A) (pr1 (iscomp2binoptransrel (pr1 R) (eqreltrans _) (pr2 R)))). unfold setquotfun2. rewrite setquotuniv2comm. apply idpath. Defined.

Lemma `commrngquotrandop2 { A : commrng } (R : @rngeqrel A) (a b : A) : (@op2 (commrngquot R)) (setquotpr (pr1 R) a) (setquotpr (pr1 R) b) ~> setquotpr (pr1 R) (a * b).` Proof. intros. change (@op2 (commrngquot R)) with (setquotfun2 R R (@op2 A) (pr2 (iscomp2binoptransrel (pr1 R) (eqreltrans _) (pr2 R)))). unfold setquotfun2. rewrite setquotuniv2comm. apply idpath. Defined.

Lemma setquotprandpadicplus (p : hz) (is : isapime p) (a b :
fpscommrng hz) : (@op1 (commrngofadicints p is)) (setquotpr (carryequiv p (isaprimetoneq0 is)) a) (setquotpr (carryequiv p (isaprimetoneq0 is)) b) \rightarrow setquotpr (carryequiv p (isaprimetoneq0 is)) (a + b). Proof. intros. apply commrngquotprandop1. Defined.

Lemma setquotprandpadictimes (p : hz) (is : isapime p) (a b :
fpscommrng hz) : (@op2 (commrngofadicints p is)) (setquotpr (carryequiv p (isaprimetoneq0 is)) a) (setquotpr (carryequiv p (isaprimetoneq0 is)) b) \rightarrow setquotpr (carryequiv p (isaprimetoneq0 is)) (a * b). Proof. intros. apply commrngquotprandop2. Defined.

Lemma padicplusisbinopapart0 (p : hz) (is : isapime p) (a b c :
fpscommrng hz) (u : padicapart0 p is (a + b) (a + c)) :
padicapart0 p is b c. Proof. intros. apply u. intros n. destruct n as [n n']. set (P := fun x : nat => neq hz (carry p (isaprimetoneq0 is) (a + b) x) (carry p (isaprimetoneq0 is) (a + c) x)). assert (isdecnatprop P) as isdec. intros m. destruct (isdeceqhz (carry p (isaprimetoneq0 is) (a + b) m) (carry p (isaprimetoneq0 is) (a + c) m)) as [l | r]. apply ii2. intros j. apply j. assumption. apply i1. assumption.

set (le := leastelementprinciple n P isdec n'). apply le. intro k. destruct k as [k k']. destruct k' as [k' k'']. destruct k. apply total2tohexists. split with 0%nat. intros j. apply k'. unfold carry. unfold precarry. change ((a + b) 0%nat) with (a 0%nat + b 0%nat). change ((a + c) 0%nat) with (a 0%nat + c 0%nat). unfold carry in j. unfold precarry in j. rewrite hzremaindermodandplus. rewrite j. rewrite <- hzremaindermodandplus. apply idpath.

destruct (isdeceqhz (carry p (isaprimetoneq0 is) b (S k)) (carry p (isaprimetoneq0 is) c (S k))) as [l | r]. apply (padicapartandplusprecarryl p is a b c k). intros j. apply k'. rewrite (carryandplus). unfold carry at 1. change (hzremaindermod p (isaprimetoneq0 is) (carry p (isaprimetoneq0 is) a (S k) + carry p (isaprimetoneq0 is) b (S k) + hzquotientmod p (isaprimetoneq0 is) (precarry p (isaprimetoneq0 is) (carry p (isaprimetoneq0 is) a + carry p (isaprimetoneq0 is) b) k)) \rightarrow carry p (isaprimetoneq0 is) (a + c) (S k)). rewrite l. rewrite j. rewrite (carryandplus p (isaprimetoneq0 is) a c). unfold carry at 5. change (precarry p (isaprimetoneq0 is) (carry p (isaprimetoneq0 is) a + carry p (isaprimetoneq0 is) c) (S k)) with (carry p (isaprimetoneq0 is) a (S k) + carry p (isaprimetoneq0 is) c (S k) + hzquotientmod p (isaprimetoneq0 is) (precarry p (isaprimetoneq0 is) (carry p (isaprimetoneq0 is) a + carry p (isaprimetoneq0 is) c) k)). apply idpath. apply total2tohexists. split with (S k). assumption. Defined.

Lemma padicplusisbinopapart1 (p : hz) (is : isapime p) :
isbinopapart1 (padicapart p is) (padicplus p is). Proof.
intros. unfold isbinopapart1. assert (forall x x' x'' :
commrngofadicints p is, isaprop ((pr1 (padicapart p is) (padicplus p is x x') (padicplus p is x x'')) \rightarrow ((pr1 (padicapart p is) (x' x'')) as int. intros. apply impred. intros. apply (pr1 (padicapart p is)). apply (setquotuniv3prop _ (fun x x' x'' => hPropair _ (int x x' x'')))). intros a b c. change (pr1 (padicapart p is) (padicplus p is (setquotpr (rngcarryequiv p (isaprimetoneq0 is)) a) (setquotpr (rngcarryequiv p (isaprimetoneq0 is)) b) (padicplus p is (setquotpr (rngcarryequiv p (isaprimetoneq0 is)) a) (setquotpr (rngcarryequiv p (isaprimetoneq0 is)) c)) \rightarrow pr1 (padicapart p is) (setquotpr (rngcarryequiv p (isaprimetoneq0 is)) b) (setquotpr (rngcarryequiv p (isaprimetoneq0 is)) c)). unfold

padicplus. rewrite 2! setquotprandpadicplus. rewrite 2!
padicapartcomputation. apply padicplusisbinopapart0. Defined.

Lemma padicplusisbinopaparttr (p : hz) (is : isapime p) :
isbinopaparttr (padicapart p is) (padicplus p is). Proof.
intros. unfold isbinopaparttr. intros a b c. unfold padicplus. rewrite (@rngcomm1 (commrngofadicints p is) b a). rewrite (@rngcomm1 (commrngofadicints p is) c a). apply padicplusisbinopapart1. Defined.

Lemma padicapartandtimestepsprecarryl (p : hz) (is : isapime p) (a b c :
fpscommrng hz) (n : nat) (x : neq _ (precarry p (isaprimetoneq0 is) (carry p (isaprimetoneq0 is) a * carry p (isaprimetoneq0 is) b) n) ((precarry p (isaprimetoneq0 is) (carry p (isaprimetoneq0 is) a * carry p (isaprimetoneq0 is) c) n)) : (padicapart0 p is) b c. Proof. intros. set (P := fun x : nat => neq hz (precarry p (isaprimetoneq0 is) (carry p (isaprimetoneq0 is) a * carry p (isaprimetoneq0 is) b) x) (precarry p (isaprimetoneq0 is) (carry p (isaprimetoneq0 is) a * carry p (isaprimetoneq0 is) c) x)). assert (isdecnatprop P) as isdec. intros m. destruct (isdeceqhz (precarry p (isaprimetoneq0 is) (carry p (isaprimetoneq0 is) a * carry p (isaprimetoneq0 is) b) m) (precarry p (isaprimetoneq0 is) (carry p (isaprimetoneq0 is) a * carry p (isaprimetoneq0 is) c) m)) as [l | r]. apply ii2. intros j. apply j. assumption. apply i1. assumption. set (leexists := leastelementprinciple n P isdec x). apply leexists. intro k. destruct k as [k k']. destruct k' as [k' k'']. induction k. apply total2tohexists. split with 0%nat. intros i. apply k'. change (carry p (isaprimetoneq0 is) a 0%nat * carry p (isaprimetoneq0 is) b 0%nat \rightarrow (carry p (isaprimetoneq0 is) a 0%nat * carry p (isaprimetoneq0 is) c 0%nat)). rewrite i. apply idpath. set (Q := (fun o : nat => hPropair (carry p (isaprimetoneq0 is) b o \rightarrow carry p (isaprimetoneq0 is) c o) (isasetzh _))). assert (isdecnatprop Q) as isdec'. intro o. destruct (isdeceqhz (carry p (isaprimetoneq0 is) b o) (carry p (isaprimetoneq0 is) c o)) as [l | r]. apply i1. assumption. apply ii2. assumption. destruct (isdecisbndqdec Q isdec' (S k)) as [l | r]. assert hfalse as xx. apply (k' k). apply natlthnsn. intro j. apply k'. change ((natsummation0 (S k) (fun x : nat => carry p (isaprimetoneq0 is) a x * carry p (isaprimetoneq0 is) b (minus (S k) x))) + hzquotientmod p (isaprimetoneq0 is) (precarry p (isaprimetoneq0 is) (carry p (isaprimetoneq0 is) a * carry p (isaprimetoneq0 is) b) k) \rightarrow ((natsummation0 (S k) (fun x : nat => carry p (isaprimetoneq0 is) a x * carry p (isaprimetoneq0 is) c (minus (S k) x))) + hzquotientmod p (isaprimetoneq0 is) (precarry p (isaprimetoneq0 is) (carry p (isaprimetoneq0 is) a * carry p (isaprimetoneq0 is) c) k)). assert (natsummation0 (S k) (fun x0 : nat => carry p (isaprimetoneq0 is) a x0 * carry p (isaprimetoneq0 is) b (minus (S k) x0)) \rightarrow natsummation0 (S k) (fun x0 : nat => carry p (isaprimetoneq0 is) a x0 * carry p (isaprimetoneq0 is) c (minus (S k) x0))) as f. apply natsummationpathsupperfixed. intros m y. rewrite (l (minus (S k) m)). apply idpath. apply minusleh. rewrite f. rewrite j. apply idpath. contradiction.

apply r. intros o. destruct o as [o o']. apply
total2tohexists. split with o. apply o'. Defined.

Lemma padictimesisbinopapart0 (p : hz) (is : isapime p) (a b c :
fpscommrng hz) (u : padicapart0 p is (a * b) (a * c)) :
padicapart0 p is b c. Proof. intros. apply u. intros n. destruct n as [n n']. destruct n. apply total2tohexists. split with 0%nat. intros j. apply n'. rewrite carryandtimes. rewrite (carryandtimes p (isaprimetoneq0 is) a c). change (hzremaindermod (isaprimetoneq0 is) (carry p (isaprimetoneq0 is) a 0%nat * carry p (isaprimetoneq0 is) b 0%nat) \rightarrow hzremaindermod p (isaprimetoneq0 is) (carry p (isaprimetoneq0 is) a 0%nat * carry p (isaprimetoneq0 is) c 0%nat)). rewrite j. apply idpath. set (Q :=

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( fun o : nat => hProppair ( carry p ( isaprimetoneq0 is ) b o ^>
  carry p ( isaprimetoneq0 is ) c o ) ( isasethz _ _ ) ), assert (
  isdecnatprop Q ) as isdec'. intro o. destruct ( isdeceqhz ( carry p (
  isaprimetoneq0 is ) b o ) ( carry p ( isaprimetoneq0 is ) c o ) ) as [
  l | r ]. apply i1i. assumption. apply i12. assumption. destruct (
  isdecisbndqdec Q isdec' ( S n ) ) as [ l | r ]. apply (
  padicapartandtimesprecaryl p is a b c n ). intros j. assert hfalse as
  xx. apply n'. rewrite carryandtimes. rewrite ( carryandtimes p (
  isaprimetoneq0 is ) a c ). change ( hzremaindermod p ( isaprimetoneq0
  is ) ( natsummation0 ( S n ) ( fun x : nat => carry p ( isaprimetoneq0
  is ) a x * carry p ( isaprimetoneq0 is ) b ( minus ( S n ) x ) ) +
  hzquotientmod p ( isaprimetoneq0 is ) ( precarry p ( isaprimetoneq0 is
  ) ( carry p ( isaprimetoneq0 is ) a * carry p ( isaprimetoneq0 is ) b
  ) n ) ) ^> ( hzremaindermod p ( isaprimetoneq0 is ) ( natsummation0 (
  S n ) ( fun x : nat => carry p ( isaprimetoneq0 is ) a x * carry p (
  isaprimetoneq0 is ) c ( minus ( S n ) x ) ) + hzquotientmod p (
  isaprimetoneq0 is ) ( precarry p ( isaprimetoneq0 is ) ( carry p (
  isaprimetoneq0 is ) a * carry p ( isaprimetoneq0 is ) c ) n ) ) ).
  rewrite j. assert ( natsummation0 ( S n ) ( fun x0 : nat => carry p (
  isaprimetoneq0 is ) a x0 * carry p ( isaprimetoneq0 is ) b ( minus ( S n
  ) x0 ) ) ^> natsummation0 ( S n ) ( fun x0 : nat => carry p (
  isaprimetoneq0 is ) a x0 * carry p ( isaprimetoneq0 is ) c ( minus ( S n
  ) x0 ) ) ) as f. apply natsummationpathsupperfixed. intros m y. rewrite
  ( l ( minus ( S n ) m ) ). apply idpath. apply minusleh. rewrite
  f. apply idpath. contradiction. apply r. intros k. destruct k as [ k
  k' ]. apply total2tohexists. split with k. apply k'. Defined.

```

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Lemma padictimesisbinopapartl ( p : hz ) ( is : isaprime p ) :
isbinopapartl ( padicapart p is ) ( padictimes p is ). Proof.
intros. unfold isbinopapartl. assert ( forall x x' x'' :
commrngofpadicints p is, isaprop ( ( pr1 ( padicapart p is ) ) (
padictimes p is x x' ) ( padictimes p is x x'' ) -> ( ( pr1 (
padicapart p is ) ) x' x'' ) ) ) as int. intros. apply
impred. intros. apply ( pr1 ( padicapart p is ) ). apply (
setquotuniv3prop _ ( fun x x' x'' => hProppair _ ( int x x' x'' ) )
). intros a b c. change ( pr1 ( padicapart p is ) ( padictimes p is
(setquotpr (carryequiv p (isaprimetoneq0 is)) a) (setquotpr
(carryequiv p (isaprimetoneq0 is)) b) (padictimes p is (setquotpr
(carryequiv p (isaprimetoneq0 is)) a) (setquotpr (carryequiv p
(isaprimetoneq0 is)) c)) -> pr1 (padicapart p is) (setquotpr
(carryequiv p (isaprimetoneq0 is)) b) (setquotpr (carryequiv p
(isaprimetoneq0 is)) c)) ). unfold padictimes. rewrite 2!
setquotprandpadictimes. rewrite 2! padicapartcomputation. intros j.
apply ( padictimesisbinopapart0 p is a b c j ). Defined.

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Lemma padictimesisbinopapartr ( p : hz ) ( is : isaprime p ) :
isbinopapartr ( padicapart p is ) ( padictimes p is ). Proof.
intros. unfold isbinopapartr. intros a b c. unfold padictimes. rewrite
( @rngcomm2 ( commrngofpadicints p is ) b a ). rewrite ( @rngcomm2 (
commrngofpadicints p is ) c a ). apply padictimesisbinopapartl.
Defined.

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Definition acomrngofpadicints ( p : hz ) ( is : isaprime p ) :
acomrng. Proof. intros. split with ( commrngofpadicints p is
). split with ( padicapart p is ). split. split. apply (
padicplussisbinopapartl p is ). apply ( padicplussisbinopapartr p is ).
split. apply ( padictimesisbinopapartl p is ). apply (
padictimesisbinopapartr p is ). Defined.

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(** IV. The apartness domain of p-adic integers and the Heyting field of p-adic numbers *)

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Lemma precarryandzeromultl ( p : hz ) ( is : isaprime p ) ( a b :
fpscommrng hz ) ( n : nat ) ( x : forall m : nat, natlth m n -> (
  carry p ( isaprimetoneq0 is ) a m ^> 0%hz ) ) : forall m : nat, natlth
m n -> precarry p ( isaprimetoneq0 is ) ( fpstimes hz ( carry p (

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isaprimetoneq0 is ) a ) ( carry p ( isaprimetoneq0 is ) b ) ) m ^>
0%hz. Proof. intros p is a b n x m y. induction m. simpl. unfold
fpstimes. simpl. rewrite ( x 0%nat y ). rewrite hzmult0x. apply
idpath. change ( natsummation0 ( S m ) ( fun z : nat => ( carry p (
isaprimetoneq0 is ) a z ) * ( carry p ( isaprimetoneq0 is ) b ( minus
( S m ) z ) ) + hzquotientmod p ( isaprimetoneq0 is ) ( precarry p (
isaprimetoneq0 is ) ( fpstimes hz ( carry p ( isaprimetoneq0 is ) a )
( carry p ( isaprimetoneq0 is ) b ) ) m ) ) ^> 0%hz ). assert ( natlth
m n ) as u. apply ( istransnatlth _ ( S m ) _ ). apply
natlthnsn. assumption. rewrite ( IHm u ). rewrite hzgrand0q. rewrite
hzplus0. assert ( natsummation0 ( S m ) ( fun z : nat => carry p (
isaprimetoneq0 is ) a z * carry p ( isaprimetoneq0 is ) b ( minus ( S m
) z ) ) ^> ( natsummation0 ( S m ) ( fun z : nat => 0%hz ) ) ) as f.
apply natsummationpathsupperfixed. intros k v. assert ( natlth k n )
as uu. apply ( natlehlthtrans _ ( S m ) _
). assumption. assumption. rewrite ( x k uu ). rewrite hzmult0x. apply
idpath. rewrite f. rewrite natsummationae0bottom. apply idpath. intros
k l. apply idpath. Defined.

```

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Lemma precarryandzeromultr ( p : hz ) ( is : isaprime p ) ( a b :
fpscommrng hz ) ( n : nat ) ( x : forall m : nat, natlth m n -> (
  carry p ( isaprimetoneq0 is ) b m ^> 0%hz ) ) : forall m : nat, natlth
m n -> precarry p ( isaprimetoneq0 is ) ( fpstimes hz ( carry p (
isaprimetoneq0 is ) a ) ( carry p ( isaprimetoneq0 is ) b ) ) m ^>
0%hz. Proof. intros p is a b n x m y. change ( fpstimes hz ( carry p (
isaprimetoneq0 is ) a ) ( carry p ( isaprimetoneq0 is ) b ) ) with ( ( carry
p ( isaprimetoneq0 is ) a ) * ( carry p ( isaprimetoneq0 is ) b ) ). rewrite (
( @rngcomm2 ( fpscommrng hz ) ) ( carry p ( isaprimetoneq0 is ) a ) (
  carry p ( isaprimetoneq0 is ) b ) ). apply ( precarryandzeromultl p is
  b a n x m y ). Defined.

```

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Lemma hzfpstimesnonzero ( a : fpscommrng hz ) ( k : nat ) ( is :
dirprod ( neq hz ( a k ) 0%hz ) ( forall m : nat, natlth m k -> ( a m
) ^> 0%hz ) ) : forall k' : nat, forall b : fpscommrng hz , forall is'
: dirprod ( neq hz ( b k' ) 0%hz ) ( forall m : nat, natlth m k' -> ( b m
) ^> 0%hz ), ( a * b ) ( k + k' ) %nat ^> ( a k ) * ( b k' ).
Proof. intros a k is k'. induction k'. intros. destruct
k. simpl. apply idpath. rewrite natplus0. change ( natsummation0 k (
fun x : nat => a x * b ( minus ( S k ) x ) ) + a ( S k ) * b ( minus (
S k ) ( S k ) ) ^> a ( S k ) * b 0%nat ). assert ( natsummation0 k (
fun x : nat => a x * b ( minus ( S k ) x ) ) ^> natsummation0 k ( fun
x : nat => 0%hz ) ) as f. apply natsummationpathsupperfixed. intros m
i. assert ( natlth m ( S k ) ) as i0. apply ( natlehlthtrans _ k _
). assumption. apply natlthnsn. rewrite ( ( pr2 is ) m i0 ). rewrite
hzmult0x. apply idpath. rewrite f. rewrite
natsummationae0bottom. rewrite hzplusl0. rewrite minusnn0. apply
idpath. intros m i. apply idpath. intros. rewrite natplussm. change
( natsummation0 ( k + k' ) %nat ( fun x : nat => a x * b ( minus ( S k
+ k' ) x ) ) + a ( S k + k' ) %nat * b ( minus ( S k + k' ) ( S k + k'
) ) ) ^> a k * b ( S k' ) ). set ( b' := fpsshift b ). rewrite
minusnn0. rewrite ( ( pr2 is' ) 0%nat ( natlehlthtrans 0 k' ( S k' ) (
natleh0n k' ) ( natlthnsn k' ) ) ). rewrite hzmultx0. rewrite
hzplus0. assert ( natsummation0 ( k + k' ) %nat ( fun x : nat => a x
* b ( minus ( S k + k' ) x ) ) ^> fpstimes hz a b' ( k + k' ) %nat ) as
f. apply natsummationpathsupperfixed. intros m v. change ( S k + k'
) %nat with ( S ( k + k' ) ). rewrite <-( pathssminus ( k + k' ) %nat m
). apply idpath. apply ( natlehlthtrans _ ( k + k' ) %nat _
). assumption. apply natlthnsn. rewrite f. apply ( IHk' b'
). split. apply is'. intros m v. unfold b'. unfold fpsshift. apply
is'. assumption. Defined.

```

```

Lemma hzfpstimeswhenzero ( a : fpscommrng hz ) ( m k : nat ) ( is : (
forall m : nat, natlth m k -> ( a m ) ^> 0%hz ) ) : forall b :
fpscommrng hz, forall k' : nat, forall is' : ( forall m : nat, natlth
m k' -> ( b m ) ^> 0%hz ), natlth m ( k + k' ) %nat -> ( a * b ) m ^>
0%hz. Proof. intros a m. induction m. intros k. intros is b k' is'

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j. change ( a 0%nat * b 0%nat ~> 0%hz ). destruct k. rewrite ( is'
0%nat j ). rewrite hzmultx0. apply idpath. assert ( natlth 0 ( S k ) )
as i. apply ( natlehlthtrans _ k _ ). apply natlehOn. apply
natlthnsn. rewrite ( is 0%nat i ). rewrite hzmult0x. apply idpath.
```

```
intros k is b k' is' j. change ( natsummation0 ( S m ) ( fun x : nat
=> a x * b ( minus ( S m ) x ) ) ~> 0%hz ). change ( natsummation0
m ( fun x : nat => a x * b ( minus ( S m ) x ) ) + a ( S m ) * b (
minus ( S m ) ( S m ) ) ~> 0%hz ). assert ( a ( S m ) * b ( minus (
S m ) ( S m ) ) ~> 0%hz ) as g. destruct k. destruct k'. assert
empty. apply ( negnatgthOn ( S m ) j ). contradiction. rewrite
minusnn0. rewrite ( is' 0%nat ( natlehlthtrans 0%nat k' ( S k' ) (
natlehOn k' ) ( natlthnsn k' ) ) ). rewrite hzmultx0. apply
idpath. destruct k'. rewrite natplus0 in j. rewrite ( is ( S m ) j
). rewrite hzmult0x. apply idpath. rewrite minusnn0. rewrite ( is'
0%nat ( natlehlthtrans 0%nat k' ( S k' ) ( natlehOn k' ) ( natlthnsn
k' ) ) ). rewrite hzmultx0. apply idpath. rewrite g. rewrite
hzplus0. set ( b' := fpsshift b ). assert ( natsummation0 m ( fun x
: nat => a x * b ( minus ( S m ) x ) ) ~> natsummation0 m ( fun x :
nat => a x * b' ( minus m x ) ) ) as f. apply
natsummationpathsupperfixed. intros n i. unfold b'. unfold
fpsshift. rewrite pathssminus. apply idpath. apply ( natlehlthtrans
_ m _ ). assumption. apply natlthnsn. rewrite f. change ( ( a * b' )
m ~> 0%hz ). assert ( natlth m ( k + k' ) ) as one. apply (
istransnatlth _ ( S m ) _ ). apply natlthnsn. assumption. destruct
k'. assert ( forall m : nat, natlth m 0%nat ~> b' m ~> 0%hz ) as
two. intros m0 j0. assert empty. apply ( negnatgthOn
m0 ). assumption. contradiction. apply ( IHm k is b' 0%nat two one
). assert ( forall m : nat, natlth m k' ~> b' m ~> 0%hz ) as
two. intros m0 j0. change ( b ( S m0 ) ~> 0%hz ). apply
is'. assumption. assert ( natlth m ( k + k' ) %nat ) as
three. rewrite natplusnm in j. apply j. apply ( IHm k is b' k' two
three ). Defined.
```

```
Lemma precarryandzeromult ( p : hz ) ( is : isapime p ) ( a b :
fpscommrng hz ) ( k k' : nat ) ( x : forall m : nat, natlth m k ~>
carry p ( isaprimetoneq0 is ) a m ~> 0%hz ) ( x' : forall m : nat,
natlth m k' ~> carry p ( isaprimetoneq0 is ) b m ~> 0%hz ) : forall m
: nat, natlth m ( k + k' ) %nat ~> precarry p ( isaprimetoneq0 is ) (
fpstimes hz ( carry p ( isaprimetoneq0 is ) a ) ( carry p (
isaprimetoneq0 is ) b ) ) m ~> 0%hz. Proof. intros p is a b k k' x
x' m i. induction m. apply ( hzfpstimeswhenzero ( carry p (
isaprimetoneq0 is ) a ) 0%nat k x ( carry p ( isaprimetoneq0 is ) b )
k' x' i ). change ( ( ( carry p ( isaprimetoneq0 is ) a ) * ( carry p
( isaprimetoneq0 is ) b ) ) ( S m ) + hzquotientmod p ( isaprimetoneq0
is ) ( precarry p ( isaprimetoneq0 is ) ( fpstimes hz ( carry p (
isaprimetoneq0 is ) a ) ( carry p ( isaprimetoneq0 is ) b ) ) m ) ~>
0%hz ). rewrite ( hzfpstimeswhenzero ( carry p ( isaprimetoneq0 is )
a ) ( S m ) k x ( carry p ( isaprimetoneq0 is ) b ) k' x' i ). rewrite
hzplusl0. assert ( natlth m ( k + k' ) %nat ) as one. apply (
istransnatlth _ ( S m ) _ ). apply natlthnsn. assumption. rewrite (
IHm one ). rewrite hzqrand0q. apply idpath. Defined.
```

```
Lemma primedivorcoprime ( p a : hz ) ( is : isapime p ) : hdisj (
hzdiv p a ) ( gcd p a ( isaprimetoneq0 is ) ~> 1 ). Proof.
intros. intros P i. apply ( pr2 is ( gcd p a ( isaprimetoneq0 is ) ) (
pr1 ( gcdiscommdiv p a ( isaprimetoneq0 is ) ) ) ). intro t. apply
i. destruct t as [ t0 | t1 ]. apply ii2. assumption. apply
iii. rewrite <- t1. exact ( pr2 ( gcdiscommdiv p a ( isaprimetoneq0
is ) ) ). Defined.
```

```
Lemma primeandtimes ( p a b : hz ) ( is : isapime p ) ( x : hzdiv p (
a * b ) ) : hdisj ( hzdiv p a ) ( hzdiv p b ). Proof. intros. apply
( primedivorcoprime p a is ). intros j. intros P i. apply i. destruct
j as [ j0 | j1 ]. apply iii. assumption. apply ii2. apply x. intro
u. destruct u as [ k u ]. unfold hzdiv0 in u. set ( cd :=
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bezoutstrong a p ( isaprimetoneq0 is ) ). destruct cd as [ cd f
]. destruct cd as [ c d ]. rewrite j1 in f. simpl in f. assert ( b ~>
( ( b * c + d * k ) * p ) ) as g. assert ( b ~> b * 1 ) as g0. rewrite
hzmultl1. apply idpath. rewrite g0. rewrite ( rngrdistr hz ( b * 1 * c
) ( d * k ) p ). assert ( b * ( c * p + d * a ) ~> ( b * 1 * c * p + d
* k * p ) ) as h. rewrite ( rngldistr hz ( c * p ) ( d * a ) b
). rewrite hzmultl1. rewrite 2! ( @rngassoc2 hz ). rewrite ( @rngcomm2
hz k p ). change ( p * k ) %hz with ( p * k ) %rng in u. rewrite
u. rewrite ( @rngcomm2 hz b ( d * a ) ). rewrite ( @rngassoc2 hz
). apply idpath. rewrite <- h. rewrite f. apply idpath. intros Q
uu. apply uu. split with ( b * c + d * k ). rewrite ( @rngcomm2 hz _ p
) in g. unfold hzdiv0. apply pathsinv0. assumption. Defined.
```

```
Lemma hzremaindermodprimeandtimes ( p : hz ) ( is : isapime p ) ( a b
: hz ) ( x : hzremaindermod p ( isaprimetoneq0 is ) ( a * b ) ~> 0 ) :
hdisj ( hzremaindermod p ( isaprimetoneq0 is ) a ~> 0 ) (
hzremaindermod p ( isaprimetoneq0 is ) b ~> 0 ). Proof.
intros. assert ( hzdiv ( a * b ) ) as i. intros P i'. apply
i'. split with ( hzquotientmod p ( isaprimetoneq0 is ) ( a * b )
). unfold hzdiv0. apply pathsinv0. rewrite <- ( hzplus0 ( p *
hzquotientmod p ( isaprimetoneq0 is ) ( a * b ) %rng ) ) %hz. change ( a * b
~> ( p * hzquotientmod p ( isaprimetoneq0 is ) ( a * b ) %rng + 0 ) %rng ).
rewrite <- x. change ( p * hzquotientmod p ( isaprimetoneq0 is ) ( a * b )
+ hzremaindermod p ( isaprimetoneq0 is ) a * b ) with ( p * hzquotientmod
p ( isaprimetoneq0 is ) ( a * b ) %rng + ( hzremaindermod p ( isaprimetoneq0
is ) a * b ) %rng ) %hz. apply ( hzdivequationmod p ( isaprimetoneq0 is )
( a * b ) ). apply ( primeandtimes p a b is i ). intro t. destruct t
as [ t0 | t1 ]. apply t0. intros k. destruct k as [ k k' ]. intros Q
j. apply j. apply iil. apply pathsinv0. apply ( hzqrtestr p (
isaprimetoneq0 is ) a k ). split. rewrite hzplus0. unfold hzdiv0 in
k'. rewrite k'. apply idpath. split. apply isreflhzleh. rewrite
hzabsvalgth0. apply ( istranshzlth _ 1 _ ). apply hzlthnsn. apply
is. apply ( istranshzlth _ 1 _ ). apply hzlthnsn. apply is. apply
t1. intros k. destruct k as [ k k' ]. intros Q j. apply j. apply
ii2. apply pathsinv0. apply ( hzqrtestr p ( isaprimetoneq0 is ) b k ).
split. rewrite hzplus0. unfold hzdiv0 in k'. rewrite k'. apply
idpath. split. apply isreflhzleh. rewrite hzabsvalgth0. apply (
istranshzlth _ 1 _ ). apply hzlthnsn. apply is. apply ( istranshzlth _
1 _ ). apply hzlthnsn. apply is. Defined.
```

```
Definition padiczero ( p : hz ) ( is : isapime p ) := @rngunel1 (
commrngofpadicints p is ).
```

```
Definition padicone ( p : hz ) ( is : isapime p ) := @rngunel2 (
commrngofpadicints p is ).
```

```
Lemma padiczerocomputation ( p : hz ) ( is : isapime p ) : padiczero
p is ~> setquotpr ( carryequiv p ( isaprimetoneq0 is ) ) ( @rngunel1 (
fpscommrng hz ) ). Proof. intros. apply idpath. Defined.
```

```
Lemma padiconecomputation ( p : hz ) ( is : isapime p ) : padicone p
is ~> setquotpr ( carryequiv p ( isaprimetoneq0 is ) ) ( @rngunel2 (
fpscommrng hz ) ). Proof. intros. apply idpath. Defined.
```

```
Lemma padicintsareintdom ( p : hz ) ( is : isapime p ) ( a b :
acomrngofpadicints p is ) : a # 0 ~> b # 0 ~> a * b # 0. Proof.
intros p is. assert ( forall a b : commrngofpadicints p is, isaprop (
( pr1 ( padicapart p is ) ) a ( padiczero p is ) ~> ( pr1 ( padicapart
p is ) ) b ( padiczero p is ) ~> ( pr1 ( padicapart p is ) ) ) (
padietimes p is a b ) ( padiczero p is ) ) ) as int. intros. apply
impred. intros. apply impred. intros. apply ( pr1 ( padicapart p is )
).
```

```
apply ( setquotuniv2prop _ ( fun x y => hProppair _ ( int x y ) )
). intros a b. change ( pr1 ( padicapart p is ) ( setquotpr ( carryequiv
p ( isaprimetoneq0 is ) ) a ) ( padiczero p is ) ~> pr1 ( padicapart p is )
```

```

(setquotpr (carryequiv p (isaprimetoneq0 is)) b) (padiczero p is) ->
pr1 (padicapart p is) (padictimes p is (setquotpr (carryequiv p
(isaprimetoneq0 is)) a) (setquotpr (carryequiv p (isaprimetoneq0
is)) b)) (padiczero p is)). unfold padictimes. rewrite
padiczerocomputation. rewrite setquotprandpadictimes. rewrite 3!
padicapartcomputation. intros i j. apply i. intros i0. destruct i0
as [ i0 i1 ]. apply j. intros j0. destruct j0 as [ j0 j1 ]. rewrite
carryandzero in i1, j1. change ( ( @rngunell1 ( fpscommrng hz ) ) i0
) with 0%hz in i1. change ( ( @rngunell1 ( fpscommrng hz ) ) j0 )
with 0%hz in j1. set ( P := fun x : nat => neq hz ( carry p (
isaprimetoneq0 is ) a x ) 0 ). set ( P' := fun x : nat => neq hz (
carry p ( isaprimetoneq0 is ) b x ) 0 ). assert ( isdecnatprop P )
as isdec1. intros m. destruct ( isdeceqhz ( carry p (
isaprimetoneq0 is ) a m ) 0%hz ) as [ l | r ]. apply ii2. intro
v. apply v. assumption. apply iii. assumption. assert ( isdecnatprop
P' ) as isdec2. intros m. destruct ( isdeceqhz ( carry p (
isaprimetoneq0 is ) b m ) 0%hz ) as [ l | r ]. apply ii2. intro
v. apply v. assumption. apply iii. assumption. set ( le1 :=
leastelementprinciple i0 P isdec1 i1 ). set ( le2 :=
leastelementprinciple j0 P' isdec2 j1 ). apply le1. intro
k. destruct k as [ k k' ]. apply le2. intro o. destruct o as [ o o'
]. apply total2tohexists. split with ( k + o )%nat.

assert ( forall m : nat, natlth m k -> carry p ( isaprimetoneq0 is )
a m ~> 0%hz ) as one. intros m m0. destruct ( isdeceqhz ( carry p (
isaprimetoneq0 is ) a m ) 0%hz ) as [ left0 | right0 ]. assumption.
assert empty. apply ( ( pr2 k' ) m m0 ). assumption. contradiction.
assert ( forall m : nat, natlth m o -> carry p ( isaprimetoneq0 is )
b m ~> 0%hz ) as two. intros m m0. destruct ( isdeceqhz ( carry p (
isaprimetoneq0 is ) b m ) 0%hz ) as [ left0 | right0 ]. assumption.
assert empty. apply ( ( pr2 o' ) m m0 ). assumption. contradiction.
assert ( dirprod ( neq hz ( carry p ( isaprimetoneq0 is ) a k ) 0%hz
) ( forall m : nat, natlth m k -> ( carry p ( isaprimetoneq0 is ) a
m ) ~> 0%hz ) ) as three. split. apply k'. assumption. assert (
dirprod ( neq hz ( carry p ( isaprimetoneq0 is ) b o ) 0%hz ) (
forall m : nat, natlth m o -> ( carry p ( isaprimetoneq0 is ) b m )
~> 0%hz ) ) as four. split. apply o'. assumption. set ( f :=
hzfpstimesnonzero ( carry p ( isaprimetoneq0 is ) a ) k three o (
carry p ( isaprimetoneq0 is ) b ) four ). rewrite
carryandzero. change ( ( @rngunell1 ( fpscommrng hz ) ) ( k + o )%nat
) with 0%hz. rewrite carryandtimes.

destruct k. destruct o. rewrite <- carryandtimes. intros v. change (
hzremaindermod p ( isaprimetoneq0 is ) ( a 0%nat * b 0%nat ) ~> 0%hz
) in v. assert hfalse. apply ( hzremaindermodprimeandtimes p is ( a
0%nat ) ( b 0%nat ) v ). intros t. destruct t as [ t0 | t1 ]. apply
( pr1 k' ). apply t0. apply ( pr1 o' ). apply t1. assumption.

intros v. unfold carry at 1 in v. change ( 0 + S o )%nat with ( S o

```

```

) in v. change ( hzremaindermod p ( isaprimetoneq0 is ) ( ( carry p
( isaprimetoneq0 is ) a * carry p ( isaprimetoneq0 is ) b ) ( S o )
+ hzquotientmod p ( isaprimetoneq0 is ) ( precarry p (
isaprimetoneq0 is ) ( carry p ( isaprimetoneq0 is ) a * carry p (
isaprimetoneq0 is ) b ) o ) ) ~> 0%hz ) in v. change ( 0 + S o
)%nat with ( S o ) in f. rewrite f in v. change ( carry p (
isaprimetoneq0 is ) a * carry p ( isaprimetoneq0 is ) b ) with (
fpstimes hz ( carry p ( isaprimetoneq0 is ) a ) ( carry p (
isaprimetoneq0 is ) b ) ) in v. rewrite ( precarryandzeromult p is
a b 0%nat ( S o ) ) in v. rewrite hzgrand0q in v. rewrite hzplusr0
in v. assert hfalse. apply ( hzremaindermodprimeandtimes p is (
carry p ( isaprimetoneq0 is ) a 0%nat ) ( carry p ( isaprimetoneq0
is ) b ( S o ) ) ). assumption. intros s. destruct s as [ l | r ].
apply k'. rewrite hzgrandcarryr. assumption. apply o'. rewrite
hzgrandcarryr. assumption. assumption. apply one. apply two. apply
natlthnsn.

```

```

intros v. unfold carry at 1 in v. change ( hzremaindermod p (
isaprimetoneq0 is ) ( ( carry p ( isaprimetoneq0 is ) a * carry p (
isaprimetoneq0 is ) b ) ( S k + o )%nat + hzquotientmod p (
isaprimetoneq0 is ) ( precarry p ( isaprimetoneq0 is ) ( carry p (
isaprimetoneq0 is ) a * carry p ( isaprimetoneq0 is ) b ) ( k + o
)%nat ) ) ~> 0%hz ) in v. rewrite f in v. change ( carry p (
isaprimetoneq0 is ) a * carry p ( isaprimetoneq0 is ) b ) with (
fpstimes hz ( carry p ( isaprimetoneq0 is ) a ) ( carry p (
isaprimetoneq0 is ) b ) ) in v. rewrite ( precarryandzeromult p is
a b ( S k ) o ) in v. rewrite hzgrand0q in v. rewrite hzplusr0 in
v. assert hfalse. apply ( hzremaindermodprimeandtimes p is ( carry p
( isaprimetoneq0 is ) a ( S k ) ) ( carry p ( isaprimetoneq0 is ) b
(o ) ) ). assumption. intros s. destruct s as [ l | r ]. apply
k'. rewrite hzgrandcarryr. assumption. apply o'. rewrite
hzgrandcarryr. assumption. assumption. apply one. apply two. apply
natlthnsn. Defined.

```

```

Definition padicintegers ( p : hz ) ( is : isapime p ) : aintdom.
Proof. intros. split with ( acommrngofpadicints p is ). split.
change ( ( pr1 ( padicapart p is ) ) ( padicone p is ) ( padiczero p
is ) ). rewrite ( padiczerocomputation p is ). rewrite (
padiconecomputation p is ). rewrite padicapartcomputation. apply
total2tohexists. split with 0%nat. unfold carry. unfold
precarry. rewrite hzgrandlr. rewrite hzgrand0r. apply isnonzerornghz.
apply padicintsareintdom. Defined.

```

```

Definition padics ( p : hz ) ( is : isapime p ) : aflld := aflldfrac (
padicintegers p is ).

```

```

Close Scope rng_scope.
(** END OF FILE*)

```

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